1

Maxwell's Equations

1.1 Maxwell's Equations

Maxwell's equations describe all (classical) electromagnetic phenomena:

$$\boxed{\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}\\ \boldsymbol{\nabla}\times\boldsymbol{H} &= \boldsymbol{J}+\frac{\partial \boldsymbol{D}}{\partial t}\\ \boldsymbol{\nabla}\cdot\boldsymbol{D} &= \rho\\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0\end{aligned}} \qquad \text{(Maxwell's equations)} \tag{1.1.1}$$

The first is *Faraday's law of induction*, the second is *Ampère's law* as amended by Maxwell to include the displacement current $\partial \boldsymbol{D}/\partial t$, the third and fourth are *Gauss' laws* for the electric and magnetic fields.

The displacement current term $\partial \boldsymbol{D}/\partial t$ in Ampère's law is essential in predicting the existence of propagating electromagnetic waves. Its role in establishing charge conservation is discussed in Sec. 1.7.

Eqs. (1.1.1) are in SI units. The quantities $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic *field intensities* and are measured in units of [volt/m] and [ampere/m], respectively. The quantities $\boldsymbol{D}$ and $\boldsymbol{B}$ are the electric and magnetic *flux densities* and are in units of [coulomb/m^2] and [weber/m^2], or [tesla]. $\boldsymbol{D}$ is also called the *electric displacement*, and $\boldsymbol{B}$, the *magnetic induction*.

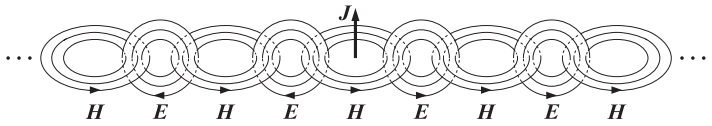
The quantities ρ and $\boldsymbol{J}$ are the *volume charge density* and *electric current density* (charge flux) of any *external* charges (that is, not including any induced polarization charges and currents.) They are measured in units of [coulomb/m^3] and [ampere/m^2]. The right-hand side of the fourth equation is zero because there are no magnetic monopole charges. Eqs. (1.3.17)–(1.3.19) display the induced polarization terms explicitly.

The charge and current densities $\rho, \boldsymbol{J}$ may be thought of as the *sources* of the electromagnetic fields. For wave propagation problems, these densities are localized in space; for example, they are restricted to flow on an antenna. The generated electric and magnetic fields are *radiated* away from these sources and can propagate to large distances to

the receiving antennas. Away from the sources, that is, in source-free regions of space, Maxwell's equations take the simpler form:

$$\boxed{\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}\\ \boldsymbol{\nabla}\times\boldsymbol{H} &= \frac{\partial \boldsymbol{D}}{\partial t}\\ \boldsymbol{\nabla}\cdot\boldsymbol{D} &= 0\\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0\end{aligned}} \qquad \text{(source-free Maxwell's equations)} \tag{1.1.2}$$

The qualitative mechanism by which Maxwell's equations give rise to propagating electromagnetic fields is shown in the figure below.



For example, a time-varying current $\boldsymbol{J}$ on a linear antenna generates a circulating and time-varying magnetic field $\boldsymbol{H}$, which through Faraday's law generates a circulating electric field $\boldsymbol{E}$, which through Ampère's law generates a magnetic field, and so on. The cross-linked electric and magnetic fields propagate away from the current source. A more precise discussion of the fields radiated by a localized current distribution is given in Chap. 14.

1.2 Lorentz Force

The force on a charge q moving with velocity $\boldsymbol{v}$ in the presence of an electric and magnetic field $\boldsymbol{E}, \boldsymbol{B}$ is called the Lorentz force and is given by:

$$\boldsymbol{F} = q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) \qquad \text{(Lorentz force)} \tag{1.2.1}$$

Newton's equation of motion is (for non-relativistic speeds):

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} = q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) \tag{1.2.2}$$

where m is the mass of the charge. The force $\boldsymbol{F}$ will increase the kinetic energy of the charge at a rate that is equal to the rate of work done by the Lorentz force on the charge, that is, $\boldsymbol{v}\cdot\boldsymbol{F}$. Indeed, the time-derivative of the kinetic energy is:

$$W_{\rm kin} = \frac{1}{2}m\,\boldsymbol{v}\cdot\boldsymbol{v} \quad\Rightarrow\quad \frac{dW_{\rm kin}}{dt} = m\,\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt} = \boldsymbol{v}\cdot\boldsymbol{F} = q\,\boldsymbol{v}\cdot\boldsymbol{E} \tag{1.2.3}$$

We note that only the electric force contributes to the increase of the kinetic energy—the magnetic force remains perpendicular to $\boldsymbol{v}$, that is, $\boldsymbol{v}\cdot(\boldsymbol{v}\times\boldsymbol{B}) = 0$.

Volume charge and current distributions $\rho, \boldsymbol{J}$ are also subjected to forces in the presence of fields. The Lorentz force *per unit volume* acting on $\rho, \boldsymbol{J}$ is given by:

$$\boldsymbol{f} = \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B} \qquad \text{(Lorentz force per unit volume)} \tag{1.2.4}$$

where $\boldsymbol{f}$ is measured in units of [newton/m^3]. If $\boldsymbol{J}$ arises from the motion of charges within the distribution ρ, then $\boldsymbol{J} = \rho\boldsymbol{v}$ (as explained in Sec. 1.6.) In this case,

$$\boldsymbol{f} = \rho\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \tag{1.2.5}$$

By analogy with Eq. (1.2.3), the quantity $\boldsymbol{v}\cdot\boldsymbol{f} = \rho\,\boldsymbol{v}\cdot\boldsymbol{E} = \boldsymbol{J}\cdot\boldsymbol{E}$ represents the *power per unit volume* of the forces acting on the moving charges, that is, the power expended by (or lost from) the fields and converted into kinetic energy of the charges, or heat. It has units of [watts/m^3]. We will denote it by:

$$\boxed{\frac{dP_{\text{loss}}}{dV} = \boldsymbol{J}\cdot\boldsymbol{E}} \qquad \text{(ohmic power losses per unit volume)} \tag{1.2.6}$$

In Sec. 1.8, we discuss its role in the conservation of energy. We will find that electromagnetic energy flowing into a region will partially increase the stored energy in that region and partially dissipate into heat according to Eq. (1.2.6).

1.3 Constitutive Relations

The electric and magnetic flux densities $\boldsymbol{D}, \boldsymbol{B}$ are related to the field intensities $\boldsymbol{E}, \boldsymbol{H}$ via the so-called *constitutive relations*, whose precise form depends on the material in which the fields exist. In *vacuum*, they take their simplest form:

$$\boxed{\begin{aligned}\boldsymbol{D} &= \epsilon_0\boldsymbol{E}\\ \boldsymbol{B} &= \mu_0\boldsymbol{H}\end{aligned}} \tag{1.3.1}$$

where ϵ_0, μ_0 are the *permittivity* and *permeability* of vacuum, with numerical values:

$$\boxed{\begin{aligned}\epsilon_0 &= 8.854\times10^{-12}\ \text{farad/m}\\ \mu_0 &= 4\pi\times10^{-7}\ \text{henry/m}\end{aligned}} \tag{1.3.2}$$

The units for ϵ_0 and μ_0 are the units of the ratios D/E and B/H, that is,

$$\frac{\text{coulomb/m}^2}{\text{volt/m}} = \frac{\text{coulomb}}{\text{volt}\cdot\text{m}} = \frac{\text{farad}}{\text{m}}\,, \qquad \frac{\text{weber/m}^2}{\text{ampere/m}} = \frac{\text{weber}}{\text{ampere}\cdot\text{m}} = \frac{\text{henry}}{\text{m}}$$

From the two quantities ϵ_0, μ_0, we can define two other physical constants, namely, the *speed of light* and the *characteristic impedance* of vacuum:

$$\boxed{c_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times10^8\ \text{m/sec}\,, \qquad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\ \text{ohm}} \tag{1.3.3}$$

The next simplest form of the constitutive relations is for simple homogeneous isotropic dielectric and for magnetic materials:

$$\boxed{\begin{aligned}\boldsymbol{D} &= \epsilon\boldsymbol{E}\\ \boldsymbol{B} &= \mu\boldsymbol{H}\end{aligned}} \tag{1.3.4}$$

These are typically valid at low frequencies. The permittivity ϵ and permeability μ are related to the *electric and magnetic susceptibilities* of the material as follows:

$$\boxed{\begin{aligned}\epsilon &= \epsilon_0(1+\chi)\\ \mu &= \mu_0(1+\chi_m)\end{aligned}} \tag{1.3.5}$$

The susceptibilities χ, χ_m are measures of the electric and magnetic polarization properties of the material. For example, we have for the electric flux density:

$$\boldsymbol{D} = \epsilon\boldsymbol{E} = \epsilon_0(1+\chi)\boldsymbol{E} = \epsilon_0\boldsymbol{E} + \epsilon_0\chi\boldsymbol{E} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \tag{1.3.6}$$

where the quantity $\boldsymbol{P} = \epsilon_0\chi\boldsymbol{E}$ represents the dielectric polarization of the material, that is, the average electric dipole moment per unit volume. In a magnetic material, we have

$$\boldsymbol{B} = \mu_0(\boldsymbol{H}+\boldsymbol{M}) = \mu_0(\boldsymbol{H}+\chi_m\boldsymbol{H}) = \mu_0(1+\chi_m)\boldsymbol{H} = \mu\boldsymbol{H} \tag{1.3.7}$$

where $\boldsymbol{M} = \chi_m\boldsymbol{H}$ is the *magnetization*, that is, the average magnetic moment per unit volume. The speed of light in the material and the characteristic impedance are:

$$c = \frac{1}{\sqrt{\mu\epsilon}}\,, \qquad \eta = \sqrt{\frac{\mu}{\epsilon}} \tag{1.3.8}$$

The *relative permittivity, permeability* and *refractive index* of a material are defined by:

$$\epsilon_{\text{rel}} = \frac{\epsilon}{\epsilon_0} = 1+\chi\,, \qquad \mu_{\text{rel}} = \frac{\mu}{\mu_0} = 1+\chi_m\,, \qquad n = \sqrt{\epsilon_{\text{rel}}\mu_{\text{rel}}} \tag{1.3.9}$$

so that $n^2 = \epsilon_{\text{rel}}\mu_{\text{rel}}$. Using the definition of Eq. (1.3.8), we may relate the speed of light and impedance of the material to the corresponding vacuum values:

$$\begin{aligned} c &= \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_0\epsilon_0\epsilon_{\text{rel}}\mu_{\text{rel}}}} = \frac{c_0}{\sqrt{\epsilon_{\text{rel}}\mu_{\text{rel}}}} = \frac{c_0}{n}\\ \eta &= \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0}}\sqrt{\frac{\mu_{\text{rel}}}{\epsilon_{\text{rel}}}} = \eta_0\sqrt{\frac{\mu_{\text{rel}}}{\epsilon_{\text{rel}}}} = \eta_0\,\frac{\mu_{\text{rel}}}{n} = \eta_0\,\frac{n}{\epsilon_{\text{rel}}}\end{aligned} \tag{1.3.10}$$

For a non-magnetic material, we have $\mu = \mu_0$, or, $\mu_{\text{rel}} = 1$, and the impedance becomes simply $\eta = \eta_0/n$, a relationship that we will use extensively in this book.

More generally, constitutive relations may be inhomogeneous, anisotropic, nonlinear, frequency dependent (dispersive), or all of the above. In *inhomogeneous materials*, the permittivity ϵ depends on the location within the material:

$$\boldsymbol{D}(\mathbf{r},t) = \epsilon(\mathbf{r})\,\boldsymbol{E}(\mathbf{r},t)$$

In *anisotropic materials*, ϵ depends on the x, y, z direction and the constitutive relations may be written component-wise in matrix (or tensor) form:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{1.3.11}$$

Anisotropy is an inherent property of the atomic/molecular structure of the dielectric. It may also be caused by the application of external fields. For example, conductors and plasmas in the presence of a constant magnetic field—such as the ionosphere in the presence of the Earth's magnetic field—become anisotropic (see for example, Problem 1.10 on the Hall effect.)

In *nonlinear materials*, ϵ may depend on the magnitude E of the applied electric field in the form:

$$\boldsymbol{D} = \epsilon(E)\boldsymbol{E}, \qquad \text{where} \qquad \epsilon(E) = \epsilon + \epsilon_2 E + \epsilon_3 E^2 + \cdots \tag{1.3.12}$$

Nonlinear effects are desirable in some applications, such as various types of electro-optic effects used in light phase modulators and phase retarders for altering polarization. In other applications, however, they are undesirable. For example, in optical fibers nonlinear effects become important if the transmitted power is increased beyond a few milliwatts. A typical consequence of nonlinearity is to cause the generation of higher harmonics, for example, if $E = E_0 e^{j\omega t}$, then Eq. (1.3.12) gives:

$$D = \epsilon(E)E = \epsilon E + \epsilon_2 E^2 + \epsilon_3 E^3 + \cdots = \epsilon E_0 e^{j\omega t} + \epsilon_2 E_0^2 e^{2j\omega t} + \epsilon_3 E_0^3 e^{3j\omega t} + \cdots$$

Thus the input frequency ω is replaced by $\omega, 2\omega, 3\omega$, and so on. In a multi-wavelength transmission system, such as a wavelength division multiplexed (WDM) optical fiber system carrying signals at closely-spaced carrier frequencies, such nonlinearities will cause the appearance of new frequencies which may be viewed as crosstalk among the original channels. For example, if the system carries frequencies ω_i, $i = 1, 2, \ldots$, then the presence of a cubic nonlinearity E^3 will cause the appearance of the frequencies $\omega_i \pm \omega_j \pm \omega_k$. In particular, the frequencies $\omega_i + \omega_j - \omega_k$ are most likely to be confused as crosstalk because of the close spacing of the carrier frequencies.

Materials with a *frequency-dependent* dielectric constant $\epsilon(\omega)$ are referred to as *dispersive*. The frequency dependence comes about because when a time-varying electric field is applied, the polarization response of the material cannot be instantaneous. Such *dynamic* response can be described by the convolutional (and causal) constitutive relationship:

$$\boldsymbol{D}(\mathbf{r}, t) = \int_{-\infty}^{t} \epsilon(t - t')\boldsymbol{E}(\mathbf{r}, t')\, dt' \tag{1.3.13}$$

which becomes multiplicative in the frequency domain:

$$\boxed{\boldsymbol{D}(\mathbf{r}, \omega) = \epsilon(\omega)\boldsymbol{E}(\mathbf{r}, \omega)} \tag{1.3.14}$$

All materials are, in fact, dispersive. However, $\epsilon(\omega)$ typically exhibits strong dependence on ω only for certain frequencies. For example, water at optical frequencies has refractive index $n = \sqrt{\epsilon_{\text{rel}}} = 1.33$, but at RF down to dc, it has $n = 9$.

In Sections 1.10–1.15, we discuss simple models of $\epsilon(\omega)$ for dielectrics, conductors, and plasmas, and clarify the nature of Ohm's law:

$$\boxed{\boldsymbol{J} = \sigma\boldsymbol{E}} \qquad \text{(Ohm's law)} \tag{1.3.15}$$

In Sec. 1.17, we discuss the Kramers-Kronig dispersion relations, which are a direct consequence of the causality of the time-domain dielectric response function $\epsilon(t)$.

One major consequence of material dispersion is *pulse spreading*, that is, the progressive widening of a pulse as it propagates through such a material. This effect limits the data rate at which pulses can be transmitted. There are other types of dispersion, such as *intermodal dispersion* in which several modes may propagate simultaneously, or *waveguide dispersion* introduced by the confining walls of a waveguide.

There exist materials that are both nonlinear and dispersive that support certain types of non-linear waves called *solitons*, in which the spreading effect of dispersion is exactly canceled by the nonlinearity. Therefore, soliton pulses maintain their shape as they propagate in such media [1177,874,875].

More complicated forms of constitutive relationships arise in chiral and gyrotropic media and are discussed in Chap. 4. The more general bi-isotropic and bi-anisotropic media are discussed in [30,95]; see also [57].

In Eqs. (1.1.1), the densities $\rho, \boldsymbol{J}$ represent the *external* or *free* charges and currents in a material medium. The induced polarization $\boldsymbol{P}$ and magnetization $\boldsymbol{M}$ may be made explicit in Maxwell's equations by using the constitutive relations:

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P}, \quad \boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) \tag{1.3.16}$$

Inserting these in Eq. (1.1.1), for example, by writing $\nabla \times \boldsymbol{B} = \mu_0 \nabla \times (\boldsymbol{H} + \boldsymbol{M}) = \mu_0(\boldsymbol{J} + \dot{\boldsymbol{D}} + \nabla \times \boldsymbol{M}) = \mu_0(\epsilon_0\dot{\boldsymbol{E}} + \boldsymbol{J} + \dot{\boldsymbol{P}} + \nabla \times \boldsymbol{M})$, we may express Maxwell's equations in terms of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \times \boldsymbol{B} &= \mu_0\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\Big[\boldsymbol{J} + \frac{\partial \boldsymbol{P}}{\partial t} + \nabla \times \boldsymbol{M}\Big] \\ \nabla \cdot \boldsymbol{E} &= \frac{1}{\epsilon_0}(\rho - \nabla \cdot \boldsymbol{P}) \\ \nabla \cdot \boldsymbol{B} &= 0 \end{aligned} \tag{1.3.17}$$

We identify the current and charge densities due to the polarization of the material as:

$$\boxed{\boldsymbol{J}_{\text{pol}} = \frac{\partial \boldsymbol{P}}{\partial t}, \quad \rho_{\text{pol}} = -\nabla \cdot \boldsymbol{P}} \qquad \text{(polarization densities)} \tag{1.3.18}$$

Similarly, the quantity $\boldsymbol{J}_{\text{mag}} = \nabla \times \boldsymbol{M}$ may be identified as the magnetization current density (note that $\rho_{\text{mag}} = 0$.) The total current and charge densities are:

$$\begin{aligned} \boldsymbol{J}_{\text{tot}} &= \boldsymbol{J} + \boldsymbol{J}_{\text{pol}} + \boldsymbol{J}_{\text{mag}} = \boldsymbol{J} + \frac{\partial \boldsymbol{P}}{\partial t} + \nabla \times \boldsymbol{M} \\ \rho_{\text{tot}} &= \rho + \rho_{\text{pol}} = \rho - \nabla \cdot \boldsymbol{P} \end{aligned} \tag{1.3.19}$$

and may be thought of as the *sources* of the fields in Eq. (1.3.17). In Sec. 14.6, we examine this interpretation further and show how it leads to the Ewald-Oseen extinction theorem and to a microscopic explanation of the origin of the refractive index.

1.4 Negative Index Media

Maxwell's equations do not preclude the possibility that one or both of the quantities ϵ, μ be negative. For example, plasmas below their plasma frequency, and metals up to optical frequencies, have $\epsilon < 0$ and $\mu > 0$, with interesting applications such as surface plasmons (see Sec. 8.5).

Isotropic media with $\mu < 0$ and $\epsilon > 0$ are more difficult to come by [153], although examples of such media have been fabricated [381].

Negative-index media, also known as left-handed media, have ϵ, μ that are simultaneously negative, $\epsilon < 0$ and $\mu < 0$. Veselago [376] was the first to study their unusual electromagnetic properties, such as having a negative index of refraction and the reversal of Snel's law.

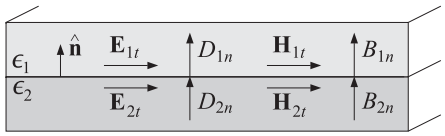
The novel properties of such media and their potential applications have generated a lot of research interest [376–457]. Examples of such media, termed "metamaterials", have been constructed using periodic arrays of wires and split-ring resonators, [382] and by transmission line elements [415–417,437,450], and have been shown to exhibit the properties predicted by Veselago.

When $\epsilon_{\rm rel} < 0$ and $\mu_{\rm rel} < 0$, the refractive index, $n^2 = \epsilon_{\rm rel}\mu_{\rm rel}$, must be defined by the negative square root $n = -\sqrt{\epsilon_{\rm rel}\mu_{\rm rel}}$. Because then $n < 0$ and $\mu_{\rm rel} < 0$ will imply that the characteristic impedance of the medium $\eta = \eta_0\mu_{\rm rel}/n$ will be positive, which as we will see later implies that the energy flux of a wave is in the same direction as the direction of propagation. We discuss such media in Sections 2.12, 7.16, and 8.6.

1.5 Boundary Conditions

The boundary conditions for the electromagnetic fields across material boundaries are given below:

$$\boxed{\begin{aligned} \boldsymbol{E}_{1t} - \boldsymbol{E}_{2t} &= 0 \\ \boldsymbol{H}_{1t} - \boldsymbol{H}_{2t} &= \boldsymbol{J}_s \times \hat{\mathbf{n}} \\ D_{1n} - D_{2n} &= \rho_s \\ B_{1n} - B_{2n} &= 0 \end{aligned}}$$



(1.5.1)

where $\hat{\mathbf{n}}$ is a unit vector normal to the boundary pointing from medium-2 into medium-1. The quantities $\rho_s, \mathbf{J}_s$ are any external *surface charge* and *surface current* densities on the boundary surface and are measured in units of [coulomb/m^2] and [ampere/m].

In words, the *tangential* components of the $\boldsymbol{E}$-field are continuous across the interface; the difference of the *tangential* components of the $\boldsymbol{H}$-field are equal to the surface

current density; the difference of the *normal* components of the flux density $\boldsymbol{D}$ are equal to the surface charge density; and the *normal* components of the magnetic flux density $\boldsymbol{B}$ are continuous.

The D_n boundary condition may also be written a form that brings out the dependence on the polarization surface charges:

$$(\epsilon_0 E_{1n} + P_{1n}) - (\epsilon_0 E_{2n} + P_{2n}) = \rho_s \quad \Rightarrow \quad \epsilon_0 (E_{1n} - E_{2n}) = \rho_s - P_{1n} + P_{2n} = \rho_{s,\rm tot}$$

The total surface charge density will be $\rho_{s,\rm tot} = \rho_s + \rho_{1s,\rm pol} + \rho_{2s,\rm pol}$, where the surface charge density of polarization charges accumulating at the surface of a dielectric is seen to be ($\hat{\mathbf{n}}$ is the outward normal from the dielectric):

$$\boxed{\rho_{s,\rm pol} = P_n = \hat{\mathbf{n}} \cdot \boldsymbol{P}} \tag{1.5.2}$$

The relative directions of the field vectors are shown in Fig. 1.5.1. Each vector may be decomposed as the sum of a part tangential to the surface and a part perpendicular to it, that is, $\boldsymbol{E} = \boldsymbol{E}_t + \boldsymbol{E}_n$. Using the vector identity,

$$\boldsymbol{E} = \hat{\mathbf{n}} \times (\boldsymbol{E} \times \hat{\mathbf{n}}) + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \boldsymbol{E}) = \boldsymbol{E}_t + \boldsymbol{E}_n \tag{1.5.3}$$

we identify these two parts as:

$$\boldsymbol{E}_t = \hat{\mathbf{n}} \times (\boldsymbol{E} \times \hat{\mathbf{n}}) \,, \qquad \boldsymbol{E}_n = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \boldsymbol{E}) = \hat{\mathbf{n}} E_n$$

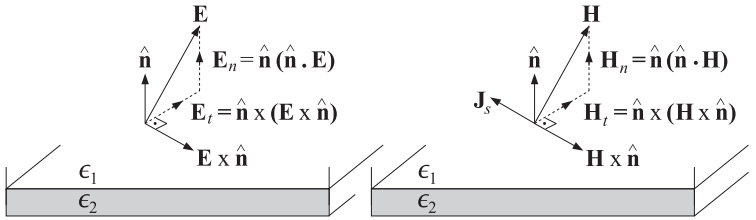


Fig. 1.5.1 Field directions at boundary.

Using these results, we can write the first two boundary conditions in the following vectorial forms, where the second form is obtained by taking the cross product of the first with $\hat{\mathbf{n}}$ and noting that $\boldsymbol{J}_s$ is purely tangential:

$$\boxed{\begin{aligned} \hat{\mathbf{n}} \times (\boldsymbol{E}_1 \times \hat{\mathbf{n}}) - \hat{\mathbf{n}} \times (\boldsymbol{E}_2 \times \hat{\mathbf{n}}) &= 0 \\ \hat{\mathbf{n}} \times (\boldsymbol{H}_1 \times \hat{\mathbf{n}}) - \hat{\mathbf{n}} \times (\boldsymbol{H}_2 \times \hat{\mathbf{n}}) &= \boldsymbol{J}_s \times \hat{\mathbf{n}} \end{aligned}} \quad \text{or,} \quad \boxed{\begin{aligned} \hat{\mathbf{n}} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) &= 0 \\ \hat{\mathbf{n}} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) &= \boldsymbol{J}_s \end{aligned}} \tag{1.5.4}$$

The boundary conditions (1.5.1) can be derived from the integrated form of Maxwell's equations if we make some additional regularity assumptions about the fields at the interfaces.

In many interface problems, there are no externally applied surface charges or currents on the boundary. In such cases, the boundary conditions may be stated as:

$$\boxed{\begin{aligned} \boldsymbol{E}_{1t} &= \boldsymbol{E}_{2t} \\ \boldsymbol{H}_{1t} &= \boldsymbol{H}_{2t} \\ D_{1n} &= D_{2n} \\ B_{1n} &= B_{2n} \end{aligned}} \qquad \text{(source-free boundary conditions)} \tag{1.5.5}$$

1.6 Currents, Fluxes, and Conservation Laws

The electric current density $\boldsymbol{J}$ is an example of a *flux vector* representing the flow of the electric charge. The concept of flux is more general and applies to any quantity that flows.[†] It could, for example, apply to energy flux, momentum flux (which translates into pressure force), mass flux, and so on.

In general, the flux of a quantity Q is defined as the amount of the quantity that flows (perpendicularly) through a unit surface in unit time. Thus, if the amount ΔQ flows through the surface ΔS in time Δt, then:

$$J = \frac{\Delta Q}{\Delta S \Delta t} \qquad \text{(definition of flux)} \tag{1.6.1}$$

When the flowing quantity Q is the electric charge, the amount of current through the surface ΔS will be $\Delta I = \Delta Q/\Delta t$, and therefore, we can write $J = \Delta I/\Delta S$, with units of [ampere/m^2].

The flux is a vectorial quantity whose direction points in the direction of flow. There is a fundamental relationship that relates the flux vector $\boldsymbol{J}$ to the transport velocity $\boldsymbol{v}$ and the volume density ρ of the flowing quantity:

$$\boxed{\boldsymbol{J} = \rho \boldsymbol{v}} \tag{1.6.2}$$

This can be derived with the help of Fig. 1.6.1. Consider a surface ΔS oriented perpendicularly to the flow velocity. In time Δt, the entire amount of the quantity contained in the cylindrical volume of height $v\Delta t$ will manage to flow through ΔS. This amount is equal to the density of the material times the cylindrical volume $\Delta V = \Delta S(v\Delta t)$, that is, $\Delta Q = \rho \Delta V = \rho\, \Delta S\, v \Delta t$. Thus, by definition:

$$J = \frac{\Delta Q}{\Delta S \Delta t} = \frac{\rho\, \Delta S\, v \Delta t}{\Delta S \Delta t} = \rho v$$

When $\boldsymbol{J}$ represents electric current density, we will see in Sec. 1.12 that Eq. (1.6.2) implies Ohm's law $\boldsymbol{J} = \sigma \boldsymbol{E}$. When the vector $\boldsymbol{J}$ represents the energy flux of a propagating electromagnetic wave and ρ the corresponding energy per unit volume, then because the speed of propagation is the velocity of light, we expect that Eq. (1.6.2) will take the form:

$$J_{\text{en}} = c\rho_{\text{en}} \tag{1.6.3}$$

[†]In this sense, the terms electric and magnetic "flux densities" for the quantities $\boldsymbol{D}, \boldsymbol{B}$ are somewhat of a misnomer because they do not represent anything that flows.

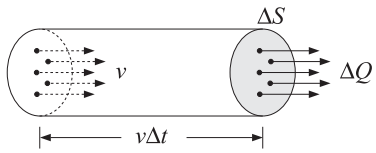


Fig. 1.6.1 Flux of a quantity.

Similarly, when J represents momentum flux, we expect to have $J_{\text{mom}} = c\rho_{\text{mom}}$. Momentum flux is defined as $J_{\text{mom}} = \Delta p/(\Delta S \Delta t) = \Delta F/\Delta S$, where p denotes momentum and $\Delta F = \Delta p/\Delta t$ is the rate of change of momentum, or the force, exerted on the surface ΔS. Thus, J_{mom} represents force per unit area, or pressure.

Electromagnetic waves incident on material surfaces exert pressure (known as radiation pressure), which can be calculated from the momentum flux vector. It can be shown that the momentum flux is numerically equal to the energy density of a wave, that is, $J_{\text{mom}} = \rho_{\text{en}}$, which implies that $\rho_{\text{en}} = \rho_{\text{mom}} c$. This is consistent with the theory of relativity, which states that the energy-momentum relationship for a photon is $E = pc$.

1.7 Charge Conservation

Maxwell added the displacement current term to Ampère's law in order to guarantee charge conservation. Indeed, taking the divergence of both sides of Ampère's law and using Gauss's law $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$, we get:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{\nabla} \cdot \boldsymbol{J} + \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{D} = \boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t}$$

Using the vector identity $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{H} = 0$, we obtain the differential form of the charge conservation law:

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0} \qquad \text{(charge conservation)} \tag{1.7.1}$$

Integrating both sides over a closed volume V surrounded by the surface S, as shown in Fig. 1.7.1, and using the divergence theorem, we obtain the integrated form of Eq. (1.7.1):

$$\oint_S \boldsymbol{J} \cdot d\boldsymbol{S} = -\frac{d}{dt} \int_V \rho\, dV \tag{1.7.2}$$

The left-hand side represents the total amount of charge flowing *outwards* through the surface S per unit time. The right-hand side represents the amount by which the charge is *decreasing* inside the volume V per unit time. In other words, charge does not disappear into (or created out of) nothingness—it decreases in a region of space only because it flows into other regions.

Another consequence of Eq. (1.7.1) is that in good conductors, there cannot be any accumulated *volume* charge. Any such charge will quickly move to the conductor's surface and distribute itself such that to make the surface into an equipotential surface.

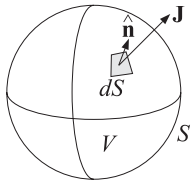


Fig. 1.7.1 Flux outwards through surface.

Assuming that inside the conductor we have $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and $\boldsymbol{J} = \sigma \boldsymbol{E}$, we obtain

$$\nabla \cdot \boldsymbol{J} = \sigma \nabla \cdot \boldsymbol{E} = \frac{\sigma}{\epsilon} \nabla \cdot \boldsymbol{D} = \frac{\sigma}{\epsilon} \rho$$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = 0 \tag{1.7.3}$$

with solution:

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r})\, e^{-\sigma t/\epsilon}$$

where $\rho_0(\mathbf{r})$ is the initial volume charge distribution. The solution shows that the volume charge disappears from inside and therefore it must accumulate on the surface of the conductor. The "relaxation" time constant $\tau_{\text{rel}} = \epsilon/\sigma$ is extremely short for good conductors. For example, in copper,

$$\tau_{\text{rel}} = \frac{\epsilon}{\sigma} = \frac{8.85 \times 10^{-12}}{5.7 \times 10^{7}} = 1.6 \times 10^{-19} \text{ sec}$$

By contrast, τ_{rel} is of the order of days in a good dielectric. For good conductors, the above argument is not quite correct because it is based on the steady-state version of Ohm's law, $\boldsymbol{J} = \sigma \boldsymbol{E}$, which must be modified to take into account the transient dynamics of the conduction charges.

It turns out that the relaxation time τ_{rel} is of the order of the collision time, which is typically 10^{-14} sec. We discuss this further in Sec. 1.13. See also Refs. [138–141].

1.8 Energy Flux and Energy Conservation

Because energy can be converted into different forms, the corresponding conservation equation (1.7.1) should have a non-zero term in the right-hand side corresponding to the rate by which energy is being lost from the fields into other forms, such as heat. Thus, we expect Eq. (1.7.1) to have the form:

$$\frac{\partial \rho_{\text{en}}}{\partial t} + \nabla \cdot \boldsymbol{J}_{\text{en}} = \text{rate of energy loss} \tag{1.8.1}$$

Assuming the ordinary constitutive relations $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$, the quantities $\rho_{\text{en}}, \boldsymbol{J}_{\text{en}}$ describing the energy density and energy flux of the fields are defined as follows,

where we introduce a change in notation:

$$\begin{aligned} \rho_{\text{en}} = w &= \frac{1}{2}\epsilon |\boldsymbol{E}|^2 + \frac{1}{2}\mu |\boldsymbol{H}|^2 = \text{energy per unit volume} \\ \boldsymbol{J}_{\text{en}} &= \boldsymbol{\mathcal{P}} = \boldsymbol{E} \times \boldsymbol{H} = \text{energy flux or Poynting vector} \end{aligned} \tag{1.8.2}$$

where $|\boldsymbol{E}|^2 = \boldsymbol{E} \cdot \boldsymbol{E}$. The quantities w and $\boldsymbol{\mathcal{P}}$ are measured in units of [joule/m^3] and [watt/m^2]. Using the identity $\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot \nabla \times \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{H}$, we find:

$$\begin{aligned} \frac{\partial w}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{P}} &= \epsilon \frac{\partial \boldsymbol{E}}{\partial t} \cdot \boldsymbol{E} + \mu \frac{\partial \boldsymbol{H}}{\partial t} \cdot \boldsymbol{H} + \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) \\ &= \frac{\partial \boldsymbol{D}}{\partial t} \cdot \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{H} + \boldsymbol{H} \cdot \nabla \times \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{H} \\ &= \left(\frac{\partial \boldsymbol{D}}{\partial t} - \nabla \times \boldsymbol{H} \right) \cdot \boldsymbol{E} + \left(\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} \right) \cdot \boldsymbol{H} \end{aligned}$$

Using Ampère's and Faraday's laws, the right-hand side becomes:

$$\boxed{\frac{\partial w}{\partial t} + \nabla \cdot \boldsymbol{\mathcal{P}} = -\boldsymbol{J} \cdot \boldsymbol{E}} \qquad \text{(energy conservation)} \tag{1.8.3}$$

As we discussed in Eq. (1.2.6), the quantity $\boldsymbol{J} \cdot \boldsymbol{E}$ represents the ohmic losses, that is, the *power per unit volume* lost into heat from the fields. The integrated form of Eq. (1.8.3) is as follows, relative to the volume and surface of Fig. 1.7.1:

$$-\oint_S \boldsymbol{\mathcal{P}} \cdot d\boldsymbol{S} = \frac{d}{dt} \int_V w\, dV + \int_V \boldsymbol{J} \cdot \boldsymbol{E} dV \tag{1.8.4}$$

It states that the total power *entering* a volume V through the surface S goes partially into increasing the field energy stored inside V and partially is lost into heat.

Example 1.8.1: Energy concepts can be used to derive the usual circuit formulas for capacitance, inductance, and resistance. Consider, for example, an ordinary plate capacitor with plates of area A separated by a distance l, and filled with a dielectric ϵ. The voltage between the plates is related to the electric field between the plates via $V = El$.

The energy density of the electric field between the plates is $w = \epsilon E^2/2$. Multiplying this by the volume between the plates, $A \cdot l$, will give the total energy stored in the capacitor. Equating this to the circuit expression $CV^2/2$, will yield the capacitance C:

$$W = \frac{1}{2}\epsilon E^2 \cdot Al = \frac{1}{2}CV^2 = \frac{1}{2}CE^2 l^2 \quad \Rightarrow \quad C = \epsilon \frac{A}{l}$$

Next, consider a solenoid with n turns wound around a cylindrical iron core of length l, cross-sectional area A, and permeability μ. The current through the solenoid wire is related to the magnetic field in the core through Ampère's law $Hl = nI$. It follows that the stored magnetic energy in the solenoid will be:

$$W = \frac{1}{2}\mu H^2 \cdot Al = \frac{1}{2}LI^2 = \frac{1}{2}L\frac{H^2 l^2}{n^2} \quad \Rightarrow \quad L = n^2 \mu \frac{A}{l}$$

Finally, consider a resistor of length l, cross-sectional area A, and conductivity σ. The voltage drop across the resistor is related to the electric field along it via $V = El$. The

current is assumed to be uniformly distributed over the cross-section A and will have density $J = \sigma E$.

The power dissipated into heat per unit volume is $JE = \sigma E^2$. Multiplying this by the resistor volume Al and equating it to the circuit expression $V^2/R = RI^2$ will give:

$$(J \cdot E)(Al) = \sigma E^2 (Al) = \frac{V^2}{R} = \frac{E^2 l^2}{R} \quad \Rightarrow \quad R = \frac{1}{\sigma}\frac{l}{A}$$

The same circuit expressions can, of course, be derived more directly using $Q = CV$, the magnetic flux $\Phi = LI$, and $V = RI$. □

Conservation laws may also be derived for the momentum carried by electromagnetic fields [41,1140]. It can be shown (see Problem 1.6) that the *momentum per unit volume* carried by the fields is given by:

$$\boxed{\boldsymbol{G} = \boldsymbol{D}\times\boldsymbol{B} = \frac{1}{c^2}\boldsymbol{E}\times\boldsymbol{H} = \frac{1}{c^2}\boldsymbol{\mathcal{P}}} \qquad \text{(momentum density)} \tag{1.8.5}$$

where we set $\boldsymbol{D} = \epsilon\boldsymbol{E}$, $\boldsymbol{B} = \mu\boldsymbol{H}$, and $c = 1/\sqrt{\epsilon\mu}$. The quantity $\boldsymbol{J}_{\text{mom}} = c\boldsymbol{G} = \boldsymbol{\mathcal{P}}/c$ will represent momentum flux, or pressure, if the fields are incident on a surface.

1.9 Harmonic Time Dependence

Maxwell's equations simplify considerably in the case of harmonic time dependence. Through the inverse Fourier transform, general solutions of Maxwell's equation can be built as linear combinations of single-frequency solutions:[†]

$$\boldsymbol{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \boldsymbol{E}(\mathbf{r},\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi} \tag{1.9.1}$$

Thus, we assume that all fields have a time dependence $e^{j\omega t}$:

$$\boldsymbol{E}(\mathbf{r},t) = \boldsymbol{E}(\mathbf{r})\, e^{j\omega t}, \quad \boldsymbol{H}(\mathbf{r},t) = \boldsymbol{H}(\mathbf{r})\, e^{j\omega t}$$

where the phasor amplitudes $\boldsymbol{E}(\mathbf{r}), \boldsymbol{H}(\mathbf{r})$ are complex-valued. Replacing time derivatives by $\partial_t \to j\omega$, we may rewrite Eq. (1.1.1) in the form:

$$\boxed{\begin{aligned} \nabla\times\boldsymbol{E} &= -j\omega\boldsymbol{B} \\ \nabla\times\boldsymbol{H} &= \boldsymbol{J} + j\omega\boldsymbol{D} \\ \nabla\cdot\boldsymbol{D} &= \rho \\ \nabla\cdot\boldsymbol{B} &= 0 \end{aligned}} \qquad \text{(Maxwell's equations)} \tag{1.9.2}$$

In this book, we will consider the solutions of Eqs. (1.9.2) in three different contexts: (a) uniform plane waves propagating in dielectrics, conductors, and birefringent media, (b) guided waves propagating in hollow waveguides, transmission lines, and optical fibers, and (c) propagating waves generated by antennas and apertures.

[†]The $e^{j\omega t}$ convention is used in the engineering literature, and $e^{-i\omega t}$ in the physics literature. One can pass from one convention to the other by making the formal substitution $j \to -i$ in all the equations.

Next, we review some conventions regarding phasors and time averages. A real-valued sinusoid has the complex phasor representation:

$$\mathcal{A}(t) = |A|\cos(\omega t + \theta) \quad \Leftrightarrow \quad A(t) = Ae^{j\omega t} \tag{1.9.3}$$

where $A = |A|e^{j\theta}$. Thus, we have $\mathcal{A}(t) = \text{Re}\big[A(t)\big] = \text{Re}\big[Ae^{j\omega t}\big]$. The time averages of the quantities $\mathcal{A}(t)$ and $A(t)$ over one period $T = 2\pi/\omega$ are zero.

The time average of the *product* of two harmonic quantities $\mathcal{A}(t) = \text{Re}\big[Ae^{j\omega t}\big]$ and $\mathcal{B}(t) = \text{Re}\big[Be^{j\omega t}\big]$ with phasors A, B is given by (see Problem 1.4):

$$\overline{\mathcal{A}(t)\mathcal{B}(t)} = \frac{1}{T}\int_0^T \mathcal{A}(t)\mathcal{B}(t)\, dt = \frac{1}{2}\text{Re}\big[AB^*\big] \tag{1.9.4}$$

In particular, the *mean-square* value is given by:

$$\overline{\mathcal{A}^2(t)} = \frac{1}{T}\int_0^T \mathcal{A}^2(t)\, dt = \frac{1}{2}\text{Re}\big[AA^*\big] = \frac{1}{2}|A|^2 \tag{1.9.5}$$

Some interesting time averages in electromagnetic wave problems are the time averages of the energy density, the Poynting vector (energy flux), and the ohmic power losses per unit volume. Using the definition (1.8.2) and the result (1.9.4), we have for these time averages:

$$\begin{aligned} w &= \frac{1}{2}\text{Re}\left[\frac{1}{2}\epsilon\boldsymbol{E}\cdot\boldsymbol{E}^* + \frac{1}{2}\mu\boldsymbol{H}\cdot\boldsymbol{H}^*\right] && \text{(energy density)} \\ \boldsymbol{\mathcal{P}} &= \frac{1}{2}\text{Re}\big[\boldsymbol{E}\times\boldsymbol{H}^*\big] && \text{(Poynting vector)} \\ \frac{dP_{\text{loss}}}{dV} &= \frac{1}{2}\text{Re}\big[\boldsymbol{J}_{\text{tot}}\cdot\boldsymbol{E}^*\big] && \text{(ohmic losses)} \end{aligned} \tag{1.9.6}$$

where $\boldsymbol{J}_{\text{tot}} = \boldsymbol{J} + j\omega\boldsymbol{D}$ is the total current in the right-hand side of Ampère's law and accounts for both conducting and dielectric losses. The time-averaged version of Poynting's theorem is discussed in Problem 1.5.

The expression (1.9.6) for the energy density w was derived under the assumption that both ϵ and μ were constants independent of frequency. In a dispersive medium, ϵ, μ become functions of frequency. In frequency bands where $\epsilon(\omega), \mu(\omega)$ are essentially real-valued, that is, where the medium is lossless, it can be shown [153] that the time-averaged energy density generalizes to:

$$w = \frac{1}{2}\text{Re}\left[\frac{1}{2}\frac{d(\omega\epsilon)}{d\omega}\boldsymbol{E}\cdot\boldsymbol{E}^* + \frac{1}{2}\frac{d(\omega\mu)}{d\omega}\boldsymbol{H}\cdot\boldsymbol{H}^*\right] \qquad \text{(lossless case)} \tag{1.9.7}$$

The derivation of (1.9.7) is as follows. Starting with Maxwell's equations (1.1.1) and without assuming any particular constitutive relations, we obtain:

$$\nabla\cdot\boldsymbol{E}\times\boldsymbol{H} = -\boldsymbol{E}\cdot\dot{\boldsymbol{D}} - \boldsymbol{H}\cdot\dot{\boldsymbol{B}} - \boldsymbol{J}\cdot\boldsymbol{E} \tag{1.9.8}$$

As in Eq. (1.8.3), we would like to interpret the first two terms in the right-hand side as the time derivative of the energy density, that is,

$$\frac{dw}{dt} = \boldsymbol{E}\cdot\dot{\boldsymbol{D}} + \boldsymbol{H}\cdot\dot{\boldsymbol{B}}$$

Anticipating a phasor-like representation, we may assume complex-valued fields and derive also the following relationship from Maxwell's equations:

$$\nabla\cdot\frac{1}{2}\,\mathrm{Re}\big[\boldsymbol{E}\times\boldsymbol{H}^*\big] = -\frac{1}{2}\,\mathrm{Re}\big[\boldsymbol{E}^*\cdot\dot{\boldsymbol{D}}\big] - \frac{1}{2}\,\mathrm{Re}\big[\boldsymbol{H}^*\cdot\dot{\boldsymbol{B}}\big] - \frac{1}{2}\,\mathrm{Re}\big[\boldsymbol{J}^*\cdot\boldsymbol{E}\big] \tag{1.9.9}$$

from which we may identify a "time-averaged" version of dw/dt,

$$\frac{d\bar{w}}{dt} = \frac{1}{2}\,\mathrm{Re}\big[\boldsymbol{E}^*\cdot\dot{\boldsymbol{D}}\big] + \frac{1}{2}\,\mathrm{Re}\big[\boldsymbol{H}^*\cdot\dot{\boldsymbol{B}}\big] \tag{1.9.10}$$

In a dispersive dielectric, the constitutive relation between $\boldsymbol{D}$ and $\boldsymbol{E}$ can be written as follows in the time and frequency domains:[†]

$$\boldsymbol{D}(t) = \int_{-\infty}^{\infty}\epsilon(t-t')\boldsymbol{E}(t')dt' \quad\Leftrightarrow\quad \boldsymbol{D}(\omega) = \epsilon(\omega)\boldsymbol{E}(\omega) \tag{1.9.11}$$

where the Fourier transforms are defined by

$$\epsilon(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\epsilon(\omega)e^{j\omega t}\,d\omega \quad\Leftrightarrow\quad \epsilon(\omega) = \int_{-\infty}^{\infty}\epsilon(t)e^{-j\omega t}dt \tag{1.9.12}$$

The time-derivative of $\boldsymbol{D}(t)$ is then

$$\dot{\boldsymbol{D}}(t) = \int_{-\infty}^{\infty}\dot{\epsilon}(t-t')\boldsymbol{E}(t')dt' \tag{1.9.13}$$

where it follows from Eq. (1.9.12) that

$$\dot{\epsilon}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}j\omega\epsilon(\omega)e^{j\omega t}d\omega \tag{1.9.14}$$

Following [153], we assume a quasi-harmonic representation for the electric field, $\boldsymbol{E}(t) = \boldsymbol{E}_0(t)\,e^{j\omega_0 t}$, where $\boldsymbol{E}_0(t)$ is a slowly-varying function of time. Equivalently, in the frequency domain we have $\boldsymbol{E}(\omega) = \boldsymbol{E}_0(\omega-\omega_0)$, assumed to be concentrated in a small neighborhood of ω_0, say, $|\omega-\omega_0|\le\Delta\omega$. Because $\epsilon(\omega)$ multiplies the narrowband function $\boldsymbol{E}(\omega)$, we may expand $\omega\epsilon(\omega)$ in a Taylor series around ω_0 and keep only the linear terms, that is, inside the integral (1.9.14), we may replace:

$$\omega\epsilon(\omega) = a_0 + b_0(\omega-\omega_0)\,,\quad a_0 = \omega_0\epsilon(\omega_0)\,,\quad b_0 = \left.\frac{d[\omega\epsilon(\omega)]}{d\omega}\right|_{\omega_0} \tag{1.9.15}$$

Inserting this into Eq. (1.9.14), we obtain the approximation

$$\dot{\epsilon}(t) \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty}[ja_0 + b_0(j\omega - j\omega_0)]e^{j\omega t}d\omega = ja_0\delta(t) + b_0(\partial_t - j\omega_0)\delta(t) \tag{1.9.16}$$

where $\delta(t)$ the Dirac delta function. This approximation is justified only insofar as it is used inside Eq. (1.9.13). Inserting (1.9.16) into Eq. (1.9.13), we find

$$\begin{aligned}\dot{\boldsymbol{D}}(t) &= \int_{-\infty}^{\infty}[ja_0\delta(t-t') + b_0(\partial_t - j\omega_0)\delta(t-t')]\boldsymbol{E}(t')dt' = \\ &= ja_0\boldsymbol{E}(t) + b_0(\partial_t - j\omega_0)\boldsymbol{E}(t) \\ &= ja_0\boldsymbol{E}_0(t)e^{j\omega_0 t} + b_0(\partial_t - j\omega_0)\left(\boldsymbol{E}_0(t)e^{j\omega_0 t}\right) \\ &= [ja_0\boldsymbol{E}_0(t) + b_0\dot{\boldsymbol{E}}_0(t)]e^{j\omega_0 t}\end{aligned} \tag{1.9.17}$$

[†]To unclutter the notation, we are suppressing the dependence on the space coordinates $\mathbf{r}$.

Because we assume that $\epsilon(\omega)$ is real (i.e., lossless) in the vicinity of ω_0, it follows that:

$$\frac{1}{2}\,\mathrm{Re}\big[\boldsymbol{E}^*\cdot\dot{\boldsymbol{D}}\big] = \frac{1}{2}\,\mathrm{Re}\big[\boldsymbol{E}_0(t)^*\cdot(ja_0\boldsymbol{E}_0(t) + b_0\dot{\boldsymbol{E}}_0(t))\big] = \frac{1}{2}b_0\,\mathrm{Re}\big[\boldsymbol{E}_0(t)^*\cdot\dot{\boldsymbol{E}}_0(t)\big]\,,\quad \text{or,}$$

$$\frac{1}{2}\,\mathrm{Re}\big[\boldsymbol{E}^*\cdot\dot{\boldsymbol{D}}\big] = \frac{d}{dt}\left[\frac{1}{4}b_0|\boldsymbol{E}_0(t)|^2\right] = \frac{d}{dt}\left[\frac{1}{4}\frac{d[\omega\epsilon(\omega)]_0}{d\omega}|\boldsymbol{E}_0(t)|^2\right] \tag{1.9.18}$$

Dropping the subscript 0, we see that the quantity under the time derivative in the right-hand side may be interpreted as a time-averaged energy density for the electric field. A similar argument can be given for the magnetic energy term of Eq. (1.9.7).

We will see in the next section that the energy density (1.9.7) consists of two parts: one part is the same as that in the vacuum case; the other part arises from the kinetic and potential energy stored in the polarizable molecules of the dielectric medium.

When Eq. (1.9.7) is applied to a plane wave propagating in a dielectric medium, one can show that (in the lossless case) the energy velocity coincides with the group velocity.

The generalization of these results to the case of a lossy medium has been studied extensively [153–167]. Eq. (1.9.7) has also been applied to the case of a "left-handed" medium in which both $\epsilon(\omega)$ and $\mu(\omega)$ are negative over certain frequency ranges. As argued by Veselago [376], such media must necessarily be dispersive in order to make Eq. (1.9.7) a positive quantity even though individually ϵ and μ are negative.

Analogous expressions to (1.9.7) may also be derived for the momentum density of a wave in a dispersive medium. In vacuum, the time-averaged momentum density is given by Eq. (1.8.5), that is,

$$\bar{\boldsymbol{G}} = \frac{1}{2}\,\mathrm{Re}\big[\epsilon_0\mu_0\,\boldsymbol{E}\times\boldsymbol{H}^*\big]$$

For the dispersive (and lossless) case this generalizes to [376,452]

$$\bar{\boldsymbol{G}} = \frac{1}{2}\,\mathrm{Re}\left[\epsilon\mu\boldsymbol{E}\times\boldsymbol{H}^* + \frac{\boldsymbol{k}}{2}\left(\frac{d\epsilon}{d\omega}|\boldsymbol{E}|^2 + \frac{d\mu}{d\omega}|\boldsymbol{H}|^2\right)\right] \tag{1.9.19}$$

1.10 Simple Models of Dielectrics, Conductors, and Plasmas

A simple model for the dielectric properties of a material is obtained by considering the motion of a bound electron in the presence of an applied electric field. As the electric field tries to separate the electron from the positively charged nucleus, it creates an electric dipole moment. Averaging this dipole moment over the volume of the material gives rise to a macroscopic dipole moment per unit volume.

A simple model for the dynamics of the displacement x of the bound electron is as follows (with $\dot{x} = dx/dt$):

$$m\ddot{x} = eE - kx - m\gamma\dot{x} \tag{1.10.1}$$

where we assumed that the electric field is acting in the x-direction and that there is a spring-like restoring force due to the binding of the electron to the nucleus, and a friction-type force proportional to the velocity of the electron.

The spring constant k is related to the resonance frequency of the spring via the relationship $\omega_0 = \sqrt{k/m}$, or, $k = m\omega_0^2$. Therefore, we may rewrite Eq. (1.10.1) as

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{e}{m}E} \tag{1.10.2}$$

The limit $\omega_0 = 0$ corresponds to unbound electrons and describes the case of good conductors. The frictional term $\gamma\dot{x}$ arises from collisions that tend to slow down the electron. The parameter γ is a measure of the rate of collisions per unit time, and therefore, $\tau = 1/\gamma$ will represent the mean-time between collisions.

In a typical conductor, τ is of the order of 10^{-14} seconds, for example, for copper, $\tau = 2.4 \times 10^{-14}$ sec and $\gamma = 4.1 \times 10^{13}$ sec^{-1}. The case of a tenuous, collisionless, plasma can be obtained in the limit $\gamma = 0$. Thus, the above simple model can describe the following cases:

a. Dielectrics, $\omega_0 \neq 0, \gamma \neq 0$.
b. Conductors, $\omega_0 = 0, \gamma \neq 0$.
c. Collisionless Plasmas, $\omega_0 = 0, \gamma = 0$.

The basic idea of this model is that the applied electric field tends to separate positive from negative charges, thus, creating an electric dipole moment. In this sense, the model contains the basic features of other types of polarization in materials, such as ionic/molecular polarization arising from the separation of positive and negative ions by the applied field, or polar materials that have a permanent dipole moment.

1.11 Dielectrics

The applied electric field $E(t)$ in Eq. (1.10.2) can have any time dependence. In particular, if we assume it is sinusoidal with frequency ω, $E(t) = Ee^{j\omega t}$, then, Eq. (1.10.2) will have the solution $x(t) = xe^{j\omega t}$, where the phasor x must satisfy:

$$-\omega^2 x + j\omega\gamma x + \omega_0^2 x = \frac{e}{m}E$$

which is obtained by replacing time derivatives by $\partial_t \to j\omega$. Its solution is:

$$x = \frac{\frac{e}{m}E}{\omega_0^2 - \omega^2 + j\omega\gamma} \tag{1.11.1}$$

The corresponding velocity of the electron will also be sinusoidal $v(t) = ve^{j\omega t}$, where $v = \dot{x} = j\omega x$. Thus, we have:

$$v = j\omega x = \frac{j\omega\frac{e}{m}E}{\omega_0^2 - \omega^2 + j\omega\gamma} \tag{1.11.2}$$

From Eqs. (1.11.1) and (1.11.2), we can find the polarization per unit volume P. We assume that there are N such elementary dipoles per unit volume. The individual electric dipole moment is $p = ex$. Therefore, the polarization per unit volume will be:

$$P = Np = Nex = \frac{\frac{Ne^2}{m}E}{\omega_0^2 - \omega^2 + j\omega\gamma} \equiv \epsilon_0\chi(\omega)E \tag{1.11.3}$$

The electric flux density will be then:

$$D = \epsilon_0 E + P = \epsilon_0\big(1 + \chi(\omega)\big)E \equiv \epsilon(\omega)E$$

where the effective permittivity $\epsilon(\omega)$ is:

$$\epsilon(\omega) = \epsilon_0 + \frac{\frac{Ne^2}{m}}{\omega_0^2 - \omega^2 + j\omega\gamma} \tag{1.11.4}$$

This can be written in a more convenient form, as follows:

$$\boxed{\epsilon(\omega) = \epsilon_0 + \frac{\epsilon_0\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma}} \tag{1.11.5}$$

where ω_p^2 is the so-called *plasma frequency* of the material defined by:

$$\boxed{\omega_p^2 = \frac{Ne^2}{\epsilon_0 m}} \quad \text{(plasma frequency)} \tag{1.11.6}$$

The model defined by (1.11.5) is known as a "Lorentz dielectric." The corresponding susceptibility, defined through $\epsilon(\omega) = \epsilon_0\big(1 + \chi(\omega)\big)$, is:

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma} \tag{1.11.7}$$

For a dielectric, we may assume $\omega_0 \neq 0$. Then, the low-frequency limit ($\omega = 0$) of Eq. (1.11.5), gives the nominal dielectric constant:

$$\epsilon(0) = \epsilon_0 + \epsilon_0\frac{\omega_p^2}{\omega_0^2} = \epsilon_0 + \frac{Ne^2}{m\omega_0^2} \tag{1.11.8}$$

The real and imaginary parts of $\epsilon(\omega)$ characterize the *refractive* and *absorptive* properties of the material. By convention, we define the imaginary part with the negative sign (because we use $e^{j\omega t}$ time dependence):

$$\epsilon(\omega) = \epsilon'(\omega) - j\epsilon''(\omega) \tag{1.11.9}$$

It follows from Eq. (1.11.5) that:

$$\epsilon'(\omega) = \epsilon_0 + \frac{\epsilon_0\omega_p^2(\omega_0^2 - \omega^2)}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}, \qquad \epsilon''(\omega) = \frac{\epsilon_0\omega_p^2\omega\gamma}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2} \tag{1.11.10}$$

Fig. 1.11.1 shows a plot of $\epsilon'(\omega)$ and $\epsilon''(\omega)$. Around the resonant frequency ω_0, the real part $\epsilon'(\omega)$ behaves in an anomalous manner, that is, it drops rapidly with frequency to values less than ϵ_0 and the material exhibits strong absorption. The term "normal dispersion" refers to an $\epsilon'(\omega)$ that is an increasing function of ω, as is the case to the far left and right of the resonant frequency.

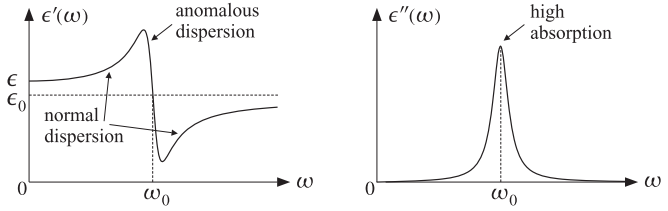


Fig. 1.11.1 Real and imaginary parts of the effective permittivity $\epsilon(\omega)$.

Real dielectric materials exhibit, of course, several such resonant frequencies corresponding to various vibrational modes and polarization mechanisms (e.g., electronic, ionic, etc.) The permittivity becomes the sum of such terms:

$$\epsilon(\omega) = \epsilon_0 + \epsilon_0 \sum_i \frac{N_i e_i^2/m_i\epsilon_0}{\omega_i^2 - \omega^2 + j\omega\gamma_i} \tag{1.11.11}$$

A more correct quantum-mechanical treatment leads essentially to the same formula:

$$\epsilon(\omega) = \epsilon_0 + \epsilon_0 \sum_{j>i} \frac{f_{ji}(N_i - N_j)e^2/m\epsilon_0}{\omega_{ji}^2 - \omega^2 + j\omega\gamma_{ji}} \tag{1.11.12}$$

where ω_{ji} are transition frequencies between energy levels, that is, $\omega_{ji} = (E_j - E_i)/\hbar$, and N_i, N_j are the populations of the lower, E_i, and upper, E_j, energy levels. The quantities f_{ji} are called "oscillator strengths." For example, for a two-level atom we have:

$$\epsilon(\omega) = \epsilon_0 + \epsilon_0 \frac{f\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma} \tag{1.11.13}$$

where we defined:

$$\omega_0 = \omega_{21}\,, \quad f = f_{21}\frac{N_1 - N_2}{N_1 + N_2}\,, \quad \omega_p^2 = \frac{(N_1 + N_2)e^2}{m\epsilon_0}$$

Normally, lower energy states are more populated, $N_i > N_j$, and the material behaves as a classical absorbing dielectric. However, if there is population inversion, $N_i < N_j$, then the corresponding permittivity term changes sign. This leads to a *negative* imaginary part, $\epsilon''(\omega)$, representing a *gain*. Fig. 1.11.2 shows the real and imaginary parts of Eq. (1.11.13) for the case of a negative effective oscillator strength $f = -1$.

The normal and anomalous dispersion bands still correspond to the bands where the real part $\epsilon'(\omega)$ is an increasing or decreasing, respectively, function of frequency. But now the normal behavior is only in the neighborhood of the resonant frequency, whereas far from it, the behavior is anomalous.

Setting $n(\omega) = \sqrt{\epsilon(\omega)/\epsilon_0}$ for the refractive index, Eq. (1.11.11) can be written in the following form, known as the Sellmeier equation (where the B_i are constants):

$$n^2(\omega) = 1 + \sum_i \frac{B_i\omega_i^2}{\omega_i^2 - \omega^2 + j\omega\gamma_i} \tag{1.11.14}$$

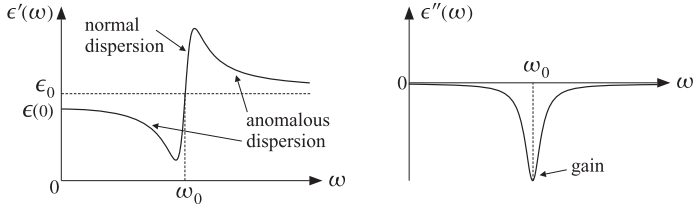


Fig. 1.11.2 Effective permittivity in a two-level gain medium with $f = -1$.

In practice, Eq. (1.11.14) is applied in frequency ranges that are far from any resonance so that one can effectively set $\gamma_i = 0$:

$$n^2(\omega) = 1 + \sum_i \frac{B_i\omega_i^2}{\omega_i^2 - \omega^2} = 1 + \sum_i \frac{B_i\lambda^2}{\lambda^2 - \lambda_i^2} \qquad \text{(Sellmeier equation)} \tag{1.11.15}$$

where λ, λ_i denote the corresponding free-space wavelengths (e.g., $\lambda = 2\pi c/\omega$). In practice, refractive index data are fitted to Eq. (1.11.15) using 2–4 terms over a desired frequency range. For example, fused silica (SiO_2) is very accurately represented over the range $0.2 \le \lambda \le 3.7$ μm by the following formula [147], where λ and λ_i are in units of μm:

$$n^2 = 1 + \frac{0.6961663\,\lambda^2}{\lambda^2 - (0.0684043)^2} + \frac{0.4079426\,\lambda^2}{\lambda^2 - (0.1162414)^2} + \frac{0.8974794\,\lambda^2}{\lambda^2 - (9.896161)^2} \tag{1.11.16}$$

1.12 Conductors

The conductivity properties of a material are described by Ohm's law, Eq. (1.3.15). To derive this law from our simple model, we use the relationship $J = \rho v$, where the volume density of the conduction charges is $\rho = Ne$. It follows from Eq. (1.11.2) that

$$J = \rho v = Nev = \frac{j\omega\dfrac{Ne^2}{m}E}{\omega_0^2 - \omega^2 + j\omega\gamma} \equiv \sigma(\omega)E$$

and therefore, we identify the conductivity $\sigma(\omega)$:

$$\sigma(\omega) = \frac{j\omega\dfrac{Ne^2}{m}}{\omega_0^2 - \omega^2 + j\omega\gamma} = \frac{j\omega\epsilon_0\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma} \tag{1.12.1}$$

We note that $\sigma(\omega)/j\omega$ is essentially the electric susceptibility considered above. Indeed, we have $J = Nev = Nej\omega x = j\omega P$, and thus, $P = J/j\omega = (\sigma(\omega)/j\omega)E$. It follows that $\epsilon(\omega) - \epsilon_0 = \sigma(\omega)/j\omega$, and

$$\epsilon(\omega) = \epsilon_0 + \frac{\epsilon_0\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma} = \epsilon_0 + \frac{\sigma(\omega)}{j\omega} \tag{1.12.2}$$

Since in a metal the conduction charges are unbound, we may take $\omega_0 = 0$ in Eq. (1.12.1). After canceling a common factor of $j\omega$, we obtain:

$$\boxed{\sigma(\omega) = \frac{\epsilon_0 \omega_p^2}{\gamma + j\omega}} \tag{1.12.3}$$

The model defined by (1.12.3) is know as the "Drude model." The nominal conductivity is obtained at the low-frequency limit, $\omega = 0$:

$$\boxed{\sigma = \frac{\epsilon_0 \omega_p^2}{\gamma} = \frac{Ne^2}{m\gamma}} \qquad \text{(nominal conductivity)} \tag{1.12.4}$$

Example 1.12.1: Copper has a mass density of 8.9×10^6 gr/m^3 and atomic weight of 63.54 (grams per mole.) Using Avogadro's number of 6×10^{23} atoms per mole, and assuming one conduction electron per atom, we find for the volume density N:

$$N = \frac{6 \times 10^{23} \ \frac{\text{atoms}}{\text{mole}}}{63.54 \ \frac{\text{gr}}{\text{mole}}} (8.9 \times 10^6 \ \frac{\text{gr}}{\text{m}^3}) (1 \ \frac{\text{electron}}{\text{atom}}) = 8.4 \times 10^{28} \ \text{electrons/m}^3$$

It follows that:

$$\sigma = \frac{Ne^2}{m\gamma} = \frac{(8.4 \times 10^{28})(1.6 \times 10^{-19})^2}{(9.1 \times 10^{-31})(4.1 \times 10^{13})} = 5.8 \times 10^7 \ \text{Siemens/m}$$

where we used $e = 1.6 \times 10^{-19}$, $m = 9.1 \times 10^{-31}$, $\gamma = 4.1 \times 10^{13}$. The plasma frequency of copper can be calculated by

$$f_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Ne^2}{m\epsilon_0}} = 2.6 \times 10^{15} \ \text{Hz}$$

which lies in the ultraviolet range. For frequencies such that $\omega \ll \gamma$, the conductivity (1.12.3) may be considered to be independent of frequency and equal to the dc value of Eq. (1.12.4). This frequency range covers most present-day RF applications. For example, assuming $\omega \leq 0.1\gamma$, we find $f \leq 0.1\gamma/2\pi = 653$ GHz. □

So far, we assumed sinusoidal time dependence and worked with the steady-state responses. Next, we discuss the transient dynamical response of a conductor subject to an arbitrary time-varying electric field $E(t)$.

Ohm's law can be expressed either in the frequency-domain or in the time-domain with the help of the Fourier transform pair of equations:

$$J(\omega) = \sigma(\omega)E(\omega) \qquad \Leftrightarrow \qquad J(t) = \int_{-\infty}^{t} \sigma(t-t')E(t')dt' \tag{1.12.5}$$

where $\sigma(t)$ is the causal inverse Fourier transform of $\sigma(\omega)$. For the simple model of Eq. (1.12.3), we have:

$$\sigma(t) = \epsilon_0 \omega_p^2 e^{-\gamma t} u(t) \tag{1.12.6}$$

where $u(t)$ is the unit-step function. As an example, suppose the electric field $E(t)$ is a constant electric field that is suddenly turned on at $t = 0$, that is, $E(t) = Eu(t)$. Then, the time response of the current will be:

$$J(t) = \int_0^t \epsilon_0 \omega_p^2 e^{-\gamma(t-t')} E dt' = \frac{\epsilon_0 \omega_p^2}{\gamma} E\left(1 - e^{-\gamma t}\right) = \sigma E\left(1 - e^{-\gamma t}\right)$$

where $\sigma = \epsilon_0 \omega_p^2/\gamma$ is the nominal conductivity of the material.

Thus, the current starts out at zero and builds up to the steady-state value of $J = \sigma E$, which is the conventional form of Ohm's law. The rise time constant is $\tau = 1/\gamma$. We saw above that τ is extremely small—of the order of 10^{-14} sec—for good conductors.

The building up of the current can also be understood in terms of the equation of motion of the conducting charges. Writing Eq. (1.10.2) in terms of the velocity of the charge, we have:

$$\dot{v}(t) + \gamma v(t) = \frac{e}{m} E(t)$$

Assuming $E(t) = Eu(t)$, we obtain the convolutional solution:

$$v(t) = \int_0^t e^{-\gamma(t-t')} \frac{e}{m} E(t')dt' = \frac{e}{m\gamma} E\left(1 - e^{-\gamma t}\right)$$

For large t, the velocity reaches the steady-state value $v_\infty = (e/m\gamma)E$, which reflects the balance between the accelerating electric field force and the retarding frictional force, that is, $m\gamma v_\infty = eE$. The quantity $e/m\gamma$ is called the *mobility* of the conduction charges. The steady-state current density results in the conventional Ohm's law:

$$J = Nev_\infty = \frac{Ne^2}{m\gamma} E = \sigma E$$

1.13 Charge Relaxation in Conductors

Next, we discuss the issue of charge relaxation in good conductors [138–141]. Writing (1.12.5) three-dimensionally and using (1.12.6), Ohm's law reads in the time domain:

$$\boldsymbol{J}(\mathbf{r}, t) = \omega_p^2 \int_{-\infty}^{t} e^{-\gamma(t-t')} \epsilon_0 \boldsymbol{E}(\mathbf{r}, t') \, dt' \tag{1.13.1}$$

Taking the divergence of both sides and using charge conservation, $\boldsymbol{\nabla} \cdot \boldsymbol{J} + \dot{\rho} = 0$, and Gauss's law, $\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho$, we obtain the following integro-differential equation for the charge density $\rho(\mathbf{r}, t)$:

$$-\dot{\rho}(\mathbf{r}, t) = \boldsymbol{\nabla} \cdot \boldsymbol{J}(\mathbf{r}, t) = \omega_p^2 \int_{-\infty}^{t} e^{-\gamma(t-t')} \epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}(\mathbf{r}, t')dt' = \omega_p^2 \int_{-\infty}^{t} e^{-\gamma(t-t')} \rho(\mathbf{r}, t')dt'$$

Differentiating both sides with respect to t, we find that ρ satisfies the second-order differential equation:

$$\ddot{\rho}(\mathbf{r}, t) + \gamma \dot{\rho}(\mathbf{r}, t) + \omega_p^2 \rho(\mathbf{r}, t) = 0 \tag{1.13.2}$$

whose solution is easily verified to be a linear combination of:

$$e^{-\gamma t/2} \cos(\omega_{\text{relax}} t) \, , \quad e^{-\gamma t/2} \sin(\omega_{\text{relax}} t) \, , \quad \text{where} \quad \omega_{\text{relax}} = \sqrt{\omega_p^2 - \frac{\gamma^2}{4}}$$

Thus, the charge density is an exponentially decaying sinusoid with a relaxation time constant that is twice the collision time $\tau = 1/\gamma$:

$$\boxed{\tau_{\text{relax}} = \frac{2}{\gamma} = 2\tau} \quad \text{(relaxation time constant)} \tag{1.13.3}$$

Typically, $\omega_p \gg \gamma$, so that ω_{relax} is practically equal to ω_p. For example, using the numerical data of Example 1.12.1, we find for copper $\tau_{\text{relax}} = 2\tau = 5\times10^{-14}$ sec. We calculate also: $f_{\text{relax}} = \omega_{\text{relax}}/2\pi = 2.6\times10^{15}$ Hz. In the limit $\gamma \to \infty$, or $\tau \to 0$, Eq. (1.13.2) reduces to the naive relaxation equation (1.7.3) (see Problem 1.9).

In addition to charge relaxation, the total relaxation time depends on the time it takes for the electric and magnetic fields to be extinguished from the inside of the conductor, as well as the time it takes for the accumulated surface charge densities to settle, the motion of the surface charges being damped because of ohmic losses. Both of these times depend on the geometry and size of the conductor [140].

1.14 Power Losses

To describe a material with *both* dielectric and conductivity properties, we may take the susceptibility to be the sum of two terms, one describing bound polarized charges and the other unbound conduction charges. Assuming different parameters $\{\omega_0, \omega_p, \gamma\}$ for each term, we obtain the total permittivity:

$$\epsilon(\omega) = \epsilon_0 + \frac{\epsilon_0\omega_{dp}^2}{\omega_{d0}^2 - \omega^2 + j\omega\gamma_d} + \frac{\epsilon_0\omega_{cp}^2}{j\omega(\gamma_c + j\omega)} \tag{1.14.1}$$

Denoting the first two terms by $\epsilon_d(\omega)$ and the third by $\sigma_c(\omega)/j\omega$, we obtain the total effective permittivity of such a material:

$$\boxed{\epsilon(\omega) = \epsilon_d(\omega) + \frac{\sigma_c(\omega)}{j\omega}} \quad \text{(effective permittivity)} \tag{1.14.2}$$

In the low-frequency limit, $\omega = 0$, the quantities $\epsilon_d(0)$ and $\sigma_c(0)$ represent the nominal dielectric constant and conductivity of the material. We note also that we can write Eq. (1.14.2) in the form:

$$j\omega\epsilon(\omega) = \sigma_c(\omega) + j\omega\epsilon_d(\omega) \tag{1.14.3}$$

These two terms characterize the relative importance of the *conduction* current and the *displacement* (polarization) current. The right-hand side in Ampère's law gives the total effective current:

$$J_{\text{tot}} = J + \frac{\partial D}{\partial t} = J + j\omega D = \sigma_c(\omega)E + j\omega\epsilon_d(\omega)E = j\omega\epsilon(\omega)E$$

where the term $J_{\text{disp}} = \partial D/\partial t = j\omega\epsilon_d(\omega)E$ represents the displacement current. The relative strength between conduction and displacement currents is the ratio:

$$\left|\frac{J_{\text{cond}}}{J_{\text{disp}}}\right| = \frac{|\sigma_c(\omega)E|}{|j\omega\epsilon_d(\omega)E|} = \frac{|\sigma_c(\omega)|}{|\omega\epsilon_d(\omega)|} \tag{1.14.4}$$

This ratio is frequency-dependent and establishes a *dividing line* between a good conductor and a good dielectric. If the ratio is much larger than unity (typically, greater than 10), the material behaves as a good conductor at that frequency; if the ratio is much smaller than one (typically, less than 0.1), then the material behaves as a good dielectric.

Example 1.14.1: This ratio can take a very wide range of values. For example, assuming a frequency of 1 GHz and using (for illustration purposes) the dc-values of the dielectric constants and conductivities, we find:

$$\left|\frac{J_{\text{cond}}}{J_{\text{disp}}}\right| = \frac{\sigma}{\omega\epsilon} = \begin{cases} 10^9 & \text{for copper with } \sigma = 5.8\times10^7 \text{ S/m and } \epsilon = \epsilon_0 \\ 1 & \text{for seawater with } \sigma = 4 \text{ S/m and } \epsilon = 72\epsilon_0 \\ 10^{-9} & \text{for a glass with } \sigma = 10^{-10} \text{ S/m and } \epsilon = 2\epsilon_0 \end{cases}$$

Thus, the ratio varies over 18 orders of magnitude! If the frequency is reduced by a factor of ten to 100 MHz, then all the ratios get multiplied by 10. In this case, seawater acts like a good conductor. □

The time-averaged ohmic power losses per unit volume within a lossy material are given by Eq. (1.9.6). Writing $\epsilon(\omega) = \epsilon'(\omega) - j\epsilon''(\omega)$, we have:

$$\boldsymbol{J}_{\text{tot}} = j\omega\epsilon(\omega)\boldsymbol{E} = j\omega\epsilon'(\omega)\boldsymbol{E} + \omega\epsilon''(\omega)\boldsymbol{E}$$

Denoting $|\boldsymbol{E}|^2 = \boldsymbol{E}\cdot\boldsymbol{E}^*$, it follows that:

$$\frac{dP_{\text{loss}}}{dV} = \frac{1}{2}\,\text{Re}\left[\boldsymbol{J}_{\text{tot}}\cdot\boldsymbol{E}^*\right] = \frac{1}{2}\omega\epsilon''(\omega)\,|\boldsymbol{E}|^2 \qquad \text{(ohmic losses)} \tag{1.14.5}$$

Writing $\epsilon_d(\omega) = \epsilon_d'(\omega) - j\epsilon_d''(\omega)$ and assuming that the conductivity $\sigma_c(\omega)$ is real-valued for the frequency range of interest (as was discussed in Example 1.12.1), we find by equating real and imaginary parts of Eq. (1.14.2):

$$\epsilon'(\omega) = \epsilon_d'(\omega)\,, \qquad \epsilon''(\omega) = \epsilon_d''(\omega) + \frac{\sigma_c(\omega)}{\omega} \tag{1.14.6}$$

Then, the power losses can be written in a form that separates the losses due to conduction and those due to the polarization properties of the dielectric:

$$\frac{dP_{\text{loss}}}{dV} = \frac{1}{2}\left(\sigma_c(\omega) + \omega\epsilon_d''(\omega)\right)|\boldsymbol{E}|^2 \qquad \text{(ohmic losses)} \tag{1.14.7}$$

A convenient way to quantify the losses is by means of the *loss tangent* defined in terms of the real and imaginary parts of the effective permittivity:

$$\boxed{\tan\theta = \frac{\epsilon''(\omega)}{\epsilon'(\omega)}} \quad \text{(loss tangent)} \tag{1.14.8}$$

where θ is the *loss angle*. Eq. (1.14.8) may be written as the sum of two loss tangents, one due to conduction and one due to polarization. Using Eq. (1.14.6), we have:

$$\tan\theta = \frac{\sigma_c(\omega)+\omega\epsilon_d''(\omega)}{\omega\epsilon_d'(\omega)} = \frac{\sigma_c(\omega)}{\omega\epsilon_d'(\omega)} + \frac{\epsilon_d''(\omega)}{\epsilon_d'(\omega)} = \tan\theta_c + \tan\theta_d \tag{1.14.9}$$

The ohmic loss per unit volume can be expressed in terms of the loss tangent as:

$$\frac{dP_{\text{loss}}}{dV} = \frac{1}{2}\omega\epsilon_d'(\omega)\tan\theta\,|\boldsymbol{E}|^2 \qquad \text{(ohmic losses)} \tag{1.14.10}$$

1.15 Plasmas

To describe a collisionless plasma, such as the ionosphere, the simple model considered in the previous sections can be specialized by choosing $\omega_0 = \gamma = 0$. Thus, the conductivity given by Eq. (1.12.3) becomes pure imaginary:

$$\sigma(\omega) = \frac{\epsilon_0\omega_p^2}{j\omega}$$

The corresponding effective permittivity of Eq. (1.12.2) becomes purely real:

$$\boxed{\epsilon(\omega) = \epsilon_0 + \frac{\sigma(\omega)}{j\omega} = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)} \tag{1.15.1}$$

The plasma frequency can be calculated from $\omega_p^2 = Ne^2/m\epsilon_0$. In the ionosphere the electron density is typically $N = 10^{12}$, which gives $f_p = 9$ MHz.

We will see in Sec. 2.6 that the propagation wavenumber of an electromagnetic wave propagating in a dielectric/conducting medium is given in terms of the effective permittivity by:

$$k = \omega\sqrt{\mu\epsilon(\omega)}$$

It follows that for a plasma:

$$k = \omega\sqrt{\mu_0\epsilon_0\left(1-\omega_p^2/\omega^2\right)} = \frac{1}{c}\sqrt{\omega^2-\omega_p^2} \tag{1.15.2}$$

where we used $c = 1/\sqrt{\mu_0\epsilon_0}$.

If $\omega > \omega_p$, the electromagnetic wave propagates without attenuation within the plasma. But if $\omega < \omega_p$, the wavenumber k becomes imaginary and the wave gets attenuated. At such frequencies, a wave incident (normally) on the ionosphere from the ground cannot penetrate and gets reflected back.

1.16 Energy Density in Lossless Dispersive Dielectrics

The lossless case is obtained from Eq. (1.11.5) by setting $\gamma = 0$, which is equivalent to assuming that ω is far from the resonance ω_0. In this case the permittivity is:

$$\epsilon(\omega) = \epsilon_0\left[1 + \frac{\omega_p^2}{\omega_0^2-\omega^2}\right]$$

from which it follows that:

$$\frac{d(\omega\epsilon)}{d\omega} = \epsilon_0\left[1 + \frac{\omega_p^2(\omega^2+\omega_0^2)}{(\omega_0^2-\omega^2)^2}\right] \tag{1.16.1}$$

Thus, the electric part of the energy density (1.9.7) will be:

$$\bar{w}_e = \frac{1}{4}\frac{d(\omega\epsilon)}{d\omega}|E|^2 = \frac{1}{4}\epsilon_0|E|^2\left[1 + \frac{\omega_p^2(\omega^2+\omega_0^2)}{(\omega_0^2-\omega^2)^2}\right] \tag{1.16.2}$$

This expression can be given a nice interpretation: The first term on the right is the energy density in vacuum and the second corresponds to the mechanical (kinetic and potential) energy of the polarization charges [154,177]. Indeed, the displacement x and velocity $v = \dot{x}$ of the polarization charges are in this case:

$$x = \frac{eE/m}{\omega_0^2-\omega^2}, \quad v = j\omega x$$

The time-averaged mechanical energy (per unit volume) is obtained by adding the kinetic and potential energies:

$$\begin{aligned}\bar{w}_{\text{mech}} &= \frac{1}{2}\,\text{Re}\left[N\left(\frac{1}{2}m|v|^2 + \frac{1}{2}m\omega_0^2|x|^2\right)\right] = \frac{1}{4}Nm(\omega^2+\omega_0^2)|x|^2 \\ &= \frac{1}{4}\frac{Nm(\omega^2+\omega_0^2)e^2|E|^2/m^2}{(\omega_0^2-\omega^2)^2} = \frac{1}{4}\epsilon_0|E|^2\left[\frac{\omega_p^2(\omega^2+\omega_0^2)}{(\omega_0^2-\omega^2)^2}\right]\end{aligned}$$

where we used the definition (1.11.6) of the plasma frequency. It follows that Eq. (1.16.2) can be written as the sum:

$$\bar{w}_e = \frac{1}{4}\frac{d(\omega\epsilon)}{d\omega}|E|^2 = \frac{1}{4}\epsilon_0|E|^2 + \bar{w}_{\text{mech}} = \bar{w}_{\text{vac}} + \bar{w}_{\text{mech}} \tag{1.16.3}$$

1.17 Kramers-Kronig Dispersion Relations

The convolutional form of Eq. (1.3.13) implies causality, that is, the value of $\boldsymbol{D}(\mathbf{r},t)$ at the present time t depends only on the past values of $\boldsymbol{E}(\mathbf{r},t')$, $t' \le t$.

This condition is equivalent to requiring that the dielectric response $\epsilon(t)$ be a right-sided (causal) function of time, that is, $\epsilon(t) = 0$ for $t < 0$. Then, Eq. (1.3.13) may be written as ordinary convolution by extending the integration range over all times:

$$\boldsymbol{D}(\mathbf{r},t) = \int_{-\infty}^{t}\epsilon(t-t')\boldsymbol{E}(\mathbf{r},t')\,dt' = \int_{-\infty}^{\infty}\epsilon(t-t')\boldsymbol{E}(\mathbf{r},t')\,dt'$$

Because $\boldsymbol{D}(\mathbf{r},t) = \epsilon_0\boldsymbol{E}(\mathbf{r},t) + \boldsymbol{P}(\mathbf{r},t)$, we may define the time-domain susceptibility function $\chi(t)$ through:

$$\epsilon(t) = \epsilon_0\delta(t) + \epsilon_0\chi(t) \tag{1.17.1}$$

where $\delta(t)$ is the Dirac delta function. Therefore, if $\epsilon(t)$ is causal, so is $\chi(t)$. The polarization is then given by:

$$\boldsymbol{P}(\mathbf{r},t) = \epsilon_0\int_{-\infty}^{t}\chi(t-t')\boldsymbol{E}(\mathbf{r},t')\,dt' = \epsilon_0\int_{-\infty}^{\infty}\chi(t-t')\boldsymbol{E}(\mathbf{r},t')\,dt' \tag{1.17.2}$$

In the frequency domain, this becomes multiplicative: $\boldsymbol{P}(\mathbf{r},\omega) = \epsilon_0\chi(\omega)\boldsymbol{E}(\mathbf{r},\omega)$. The Kramers-Kronig relations are the frequency-domain expression of causality and relate the real and imaginary parts of the susceptibility function $\chi(\omega)$. Here, the functions $\chi(t)$ and $\chi(\omega)$ are Fourier transform pairs:

$$\chi(\omega) = \int_{-\infty}^{\infty} \chi(t)\, e^{-j\omega t} dt \quad \Leftrightarrow \quad \chi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\omega)\, e^{j\omega t} d\omega \tag{1.17.3}$$

The causality condition, $\chi(t) = 0$ for $t < 0$, can be expressed in terms of the unit-step function $u(t)$ in the equivalent manner:

$$\chi(t) = \chi(t)\, u(t)\,, \quad \text{for all } t \tag{1.17.4}$$

Using the property that the Fourier transform of a product of two time functions is the convolution of their Fourier transforms, it follows that Eq. (1.17.4) can be written in the equivalent frequency-domain form:

$$\chi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\omega')\, U(\omega - \omega')\, d\omega' \tag{1.17.5}$$

where $U(\omega)$ is the Fourier transform of the unit-step. Eq. (1.17.5) is essentially the Kramers-Kronig relation. The function $U(\omega)$ is given by the well-known expression:

$$U(\omega) = \lim_{\epsilon\to 0+} \frac{1}{j\omega + \epsilon} = \mathcal{P}\frac{1}{j\omega} + \pi\delta(\omega) \tag{1.17.6}$$

where $\mathcal{P}$ denotes the "principal value." Inserting (1.17.6) into (1.17.5), we have:

$$\begin{aligned}\chi(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\omega')\left[\mathcal{P}\frac{1}{j(\omega-\omega')} + \pi\delta(\omega-\omega')\right] d\omega' \\ &= \frac{1}{2\pi j}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega-\omega'} d\omega' + \frac{1}{2}\chi(\omega)\end{aligned}$$

Rearranging terms and canceling a factor of $1/2$, we obtain the Kramers-Kronig relation in its complex-valued form:[†]

$$\chi(\omega) = \frac{1}{\pi j}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega-\omega'} d\omega' \qquad \text{(Kramers-Kronig)} \tag{1.17.7}$$

The reason for applying this relation to $\chi(\omega)$ instead of $\epsilon(\omega)$ is that $\chi(\omega)$ falls off sufficiently fast for large ω to make the integral in (1.17.5) convergent, whereas $\epsilon(\omega)$ tends to the constant ϵ_0.

Setting $\chi(\omega) = \chi_r(\omega) - j\chi_i(\omega)$ and separating (1.17.7) into its real and imaginary parts, we obtain the conventional form of the Kramers-Kronig dispersion relations:

$$\boxed{\begin{aligned}\chi_r(\omega) &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi_i(\omega')}{\omega'-\omega} d\omega' \\ \chi_i(\omega) &= -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi_r(\omega')}{\omega'-\omega} d\omega'\end{aligned}} \qquad \text{(Kramers-Kronig relations)} \tag{1.17.8}$$

[†]The right-hand side (without the j) in (1.17.7) is known as a Hilbert transform. Exchanging the roles of t and ω, such transforms, known also as 90^0 phase shifters, are used widely in signal processing for generating single-sideband communications signals.

Because the time-response $\chi(t)$ is real-valued, its Fourier transform $\chi(\omega)$ will satisfy the Hermitian symmetry property $\chi(-\omega) = \chi^*(\omega)$, which is equivalent to the even symmetry of its real part, $\chi_r(-\omega) = \chi_r(\omega)$, and the odd symmetry of its imaginary part, $\chi_i(-\omega) = -\chi_i(\omega)$. Taking advantage of these symmetries, the range of integration in (1.17.8) can be folded in half resulting in:

$$\begin{aligned}\chi_r(\omega) &= \frac{2}{\pi}\mathcal{P}\int_0^{\infty} \frac{\omega'\chi_i(\omega')}{\omega'^2-\omega^2} d\omega' \\ \chi_i(\omega) &= -\frac{2}{\pi}\mathcal{P}\int_0^{\infty} \frac{\omega\chi_r(\omega')}{\omega'^2-\omega^2} d\omega'\end{aligned} \tag{1.17.9}$$

There are several other ways to prove the Kramers-Kronig relations. For example, a more direct way is to state the causality condition in terms of the signum function $\text{sign}(t)$. Indeed, because $u(t) = \big(1 + \text{sign}(t)\big)/2$, Eq. (1.17.4) may be written in the equivalent form $\chi(t) = \chi(t)\,\text{sign}(t)$. Then, Eq. (1.17.7) follows by applying the same frequency-domain convolution argument using the Fourier transform pair:

$$\text{sign}(t) \quad \Leftrightarrow \quad \mathcal{P}\frac{2}{j\omega} \tag{1.17.10}$$

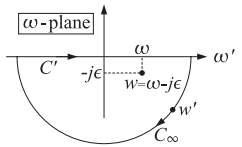
Alternatively, the causality condition can be expressed as $u(-t)\chi(t) = 0$. This approach is explored in Problem 1.12. Another proof is based on the *analyticity* properties of $\chi(\omega)$. Because of the causality condition, the Fourier integral in (1.17.3) can be restricted to the time range $0 < t < \infty$:

$$\chi(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\chi(t)\, dt = \int_0^{\infty} e^{-j\omega t}\chi(t)\, dt \tag{1.17.11}$$

This implies that $\chi(\omega)$ can be analytically continued into the *lower* half ω-plane, so that replacing ω by $w = \omega - j\alpha$ with $\alpha \geq 0$ still gives a convergent Fourier integral in Eq. (1.17.11). Any singularities in $\chi(\omega)$ lie in the upper-half plane. For example, the simple model of Eq. (1.11.7) has poles at $\omega = \pm\bar{\omega}_0 + j\gamma/2$, where $\bar{\omega}_0 = \sqrt{\omega_0^2 - \gamma^2/4}$.

Next, we consider a clockwise closed contour $C = C' + C_\infty$ consisting of the real axis C' and an infinite semicircle C_∞ in the lower half-plane. Because $\chi(\omega)$ is analytic in the region enclosed by C, Cauchy's integral theorem implies that for any point w enclosed by C, that is, lying in the lower half-plane, we must have:

$$\chi(w) = -\frac{1}{2\pi j}\oint_C \frac{\chi(w')}{w'-w}\, dw' \tag{1.17.12}$$



where the overall minus sign arises because C was taken to be clockwise. Assuming that $\chi(\omega)$ falls off sufficiently fast for large ω, the contribution of the infinite semicircle can be ignored, thus leaving only the integral over the real axis. Setting $w = \omega - j\epsilon$ and taking the limit $\epsilon \to 0+$, we obtain the identical relationship to Eq. (1.17.5):

$$\chi(\omega) = -\lim_{\epsilon\to 0+}\frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega'-\omega+j\epsilon} d\omega' = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\omega') \lim_{\epsilon\to 0+}\frac{1}{j(\omega-\omega')+\epsilon} d\omega'$$

An interesting consequence of the Kramers-Kronig relations is that there cannot exist a dielectric medium that is purely lossless, that is, such that $\chi_i(\omega)=0$ for all ω, because this would also require that $\chi_r(\omega)=0$ for all ω.

However, in all materials, $\chi_i(\omega)$ is significantly non-zero only in the neighborhoods of the medium's resonant frequencies, as for example in Fig. 1.11.1. In the frequency bands that are sufficiently far from the resonant bands, $\chi_i(\omega)$ may be assumed to be essentially zero. Such frequency bands are called *transparency bands* [153].

1.18 Group Velocity, Energy Velocity

Assuming a nonmagnetic material ($\mu=\mu_0$), a complex-valued refractive index may be defined by:

$$n(\omega)=n_r(\omega)-jn_i(\omega)=\sqrt{1+\chi(\omega)}=\sqrt{\frac{\epsilon(\omega)}{\epsilon_0}} \tag{1.18.1}$$

where n_r, n_i are its real and imaginary parts. Setting $\chi=\chi_r-j\chi_i$ we have the condition $n_r-jn_i=\sqrt{1+\chi_r-j\chi_i}$. Upon squaring, this splits into the two real-valued equations $n_r^2-n_i^2=1+\chi_r$ and $2n_rn_i=\chi_i$, with solutions:

$$\begin{aligned} n_r &= \left[\frac{\sqrt{(1+\chi_r)^2+\chi_i^2}+(1+\chi_r)}{2}\right]^{1/2} \\ n_i &= \text{sign}(\chi_i)\left[\frac{\sqrt{(1+\chi_r)^2+\chi_i^2}-(1+\chi_r)}{2}\right]^{1/2} = \frac{\chi_i}{2n_r} \end{aligned} \tag{1.18.2}$$

This form preserves the sign of χ_i, that is, n_i and χ_i are both positive for absorbing media, or both negative for gain media. The following approximate solution is often used, which can be justified whenever $|\chi|\ll 1$ (for example, in gases):

$$n_r(\omega)-jn_r(\omega)=\sqrt{1+\chi(\omega)}\simeq 1+\frac{\chi}{2} \quad\Rightarrow\quad n_r=1+\frac{1}{2}\chi_r\,,\quad n_i=\frac{1}{2}\chi_i \tag{1.18.3}$$

We will see in Chap. 2 that a single-frequency uniform plane wave propagating, say, in the positive z-direction, has a wavenumber $k=\omega n/c=\omega(n_r-jn_i)/c\equiv\beta-j\alpha$, where c is the speed of light in vacuum. Therefore, the wave will have a space-time dependence:

$$e^{j(\omega t-kz)}=e^{j(\omega t-(\beta-j\alpha)z)}=e^{-\alpha z}e^{j(\omega t-\beta z)}=e^{-\omega n_i z/c}e^{j\omega(t-n_r z/c)} \tag{1.18.4}$$

The real part n_r defines the *phase velocity* of the wave, $v_p=\omega/\beta=c/n_r$, whereas the imaginary part n_i, or $\alpha=\omega n_i/c$, corresponds to attenuation or gain depending on the sign of n_i or χ_i.

When several such plane waves are superimposed to form a propagating pulse, we will see in Sec. 3.5 that the peak of the pulse (i.e., the point on the pulse where all the individual frequency components add up in phase), propagates with the so-called *group velocity* defined by:

$$v_g=\frac{d\omega}{d\beta}=\frac{1}{\dfrac{d\beta}{d\omega}}=\frac{c}{\dfrac{d(\omega n_r)}{d\omega}}=\frac{c}{n_r+\omega\dfrac{dn_r}{d\omega}}=\text{group velocity} \tag{1.18.5}$$

A *group refractive index* may be defined through $v_g=c/n_g$, or, $n_g=c/v_g$:

$$n_g=\frac{d(\omega n_r)}{d\omega}=n_r+\omega\frac{dn_r}{d\omega}=n_r-\lambda\frac{dn_r}{d\lambda}=\text{group refractive index} \tag{1.18.6}$$

where λ is the free-space wavelength related to ω by $\lambda=2\pi c/\omega$, and we used the differentiation property that $\omega d/d\omega=-\lambda d/d\lambda$.

Within an *anomalous* dispersion region, n_r is decreasing rapidly with ω, that is, $dn_r/d\omega<0$, as in Fig. 1.11.1. This results in a group velocity v_g, given by Eq. (1.18.5), that may be larger than c or even negative. Such velocities are called "superluminal." Light pulses propagating at superluminal group velocities are referred to as "fast light" and we discuss them further in Sec. 3.9.

Within a *normal* dispersion region (e.g., to the far left and far right of the resonant frequency ω_0 in Fig. 1.11.1), n_r is an increasing function of ω, $dn_r/d\omega>0$, which results in $v_g<c$. In specially engineered materials such as those exhibiting "electromagnetically induced transparency," the slope $dn_r/d\omega$ may be made so steep that the resulting group velocity v_g becomes extremely small, $v_g\ll c$. This is referred to as "slow light."

We close this section by showing that for lossless dispersive media, the energy velocity of a plane wave is equal to the group velocity defined by (1.18.5). This result is quite general, regardless of the frequency dependence of $\epsilon(\omega)$ and $\mu(\omega)$ (as long as these quantities are real.)

We will see in the next chapter that a plane wave propagating along the z-direction has electric and magnetic fields that are transverse to the z-direction and are related by:

$$|\boldsymbol{H}|=\frac{1}{\eta}|\boldsymbol{E}|\,,\quad \eta=\sqrt{\frac{\mu}{\epsilon}}$$

Moreover the time-averaged energy flux (in the z-direction) and energy density are:

$$\bar{\mathcal{P}}_z=\frac{|\boldsymbol{E}|^2}{2\eta}\,,\quad \bar{w}=\frac{1}{4}\frac{d(\omega\epsilon)}{d\omega}|\boldsymbol{E}|^2+\frac{1}{4}\frac{d(\omega\mu)}{d\omega}|\boldsymbol{H}|^2=\frac{1}{4}\left[\frac{d(\omega\epsilon)}{d\omega}+\frac{1}{\eta^2}\frac{d(\omega\mu)}{d\omega}\right]|\boldsymbol{E}|^2$$

The energy velocity is defined by $v_{\text{en}}=\bar{\mathcal{P}}_z/\bar{w}$. Thus, we have:

$$v_{\text{en}}^{-1}=\frac{\bar{w}}{\bar{\mathcal{P}}_z}=\frac{1}{2}\left[\eta\frac{d(\omega\epsilon)}{d\omega}+\frac{1}{\eta}\frac{d(\omega\mu)}{d\omega}\right]=\frac{1}{2}\left[\sqrt{\frac{\mu}{\epsilon}}\frac{d(\omega\epsilon)}{d\omega}+\sqrt{\frac{\epsilon}{\mu}}\frac{d(\omega\mu)}{d\omega}\right]$$

It is easily verified that the right-hand side can be expressed in terms of the wavenumber $k=\omega\sqrt{\epsilon\mu}$ in the form:

$$v_{\text{en}}^{-1}=\frac{1}{2}\left[\sqrt{\frac{\mu}{\epsilon}}\frac{d(\omega\epsilon)}{d\omega}+\sqrt{\frac{\epsilon}{\mu}}\frac{d(\omega\mu)}{d\omega}\right]=\frac{d(\omega\sqrt{\epsilon\mu})}{d\omega}=\frac{dk}{d\omega}=v_g^{-1} \tag{1.18.7}$$

which shows the equality of the energy and group velocities. See Refs. [153–167] for further discussion on this topic.

Eq. (1.18.7) is also valid for the case of lossless negative-index media and implies that the group velocity, and hence the group refractive index $n_g=c_0/v_g$, will be positive,

even though the refractive index n is negative. Writing $\epsilon = -|\epsilon|$ and $\mu = -|\mu|$ in this case and noting that $\eta = \sqrt{|\mu|/|\epsilon|}$ and $n = -\sqrt{|\epsilon\mu|}/\sqrt{\epsilon_0\mu_0}$, and $k = \omega n/c_0$, we have:

$$v_{\text{en}}^{-1} = \frac{1}{2}\left[\sqrt{\frac{|\mu|}{|\epsilon|}}\frac{d(\omega\epsilon)}{d\omega} + \sqrt{\frac{|\epsilon|}{|\mu|}}\frac{d(\omega\mu)}{d\omega}\right] = -\frac{1}{2}\left[\sqrt{\frac{|\mu|}{|\epsilon|}}\frac{d(\omega|\epsilon|)}{d\omega} + \sqrt{\frac{|\epsilon|}{|\mu|}}\frac{d(\omega|\mu|)}{d\omega}\right]$$
$$= -\frac{d(\omega\sqrt{|\epsilon\mu|})}{d\omega} = \frac{1}{c_0}\frac{d(\omega n)}{d\omega} = \frac{dk}{d\omega} = v_g^{-1}$$

from which we also obtain the usual relationship $n_g = d(\omega n)/d\omega$. The positivity of v_g and n_g follows from the positivity of the derivatives $d(\omega\epsilon)/d\omega$ and $d(\omega\mu)/d\omega$, as required to keep v_{en} positive in negative-index media [376].

1.19 Problems

1.1 Prove the vector algebra identities:

$$\begin{aligned}
&\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B}) && \text{(BAC-CAB identity)}\\
&\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}) &&\\
&|\boldsymbol{A}\times\boldsymbol{B}|^2 + |\boldsymbol{A}\cdot\boldsymbol{B}|^2 = |\boldsymbol{A}|^2|\boldsymbol{B}|^2 &&\\
&\boldsymbol{A} = \hat{\mathbf{n}}\times\boldsymbol{A}\times\hat{\mathbf{n}} + (\hat{\mathbf{n}}\cdot\boldsymbol{A})\hat{\mathbf{n}} && (\hat{\mathbf{n}}\text{ is any unit vector})
\end{aligned}$$

In the last identity, does it a make a difference whether $\hat{\mathbf{n}}\times\boldsymbol{A}\times\hat{\mathbf{n}}$ is taken to mean $\hat{\mathbf{n}}\times(\boldsymbol{A}\times\hat{\mathbf{n}})$ or $(\hat{\mathbf{n}}\times\boldsymbol{A})\times\hat{\mathbf{n}}$?

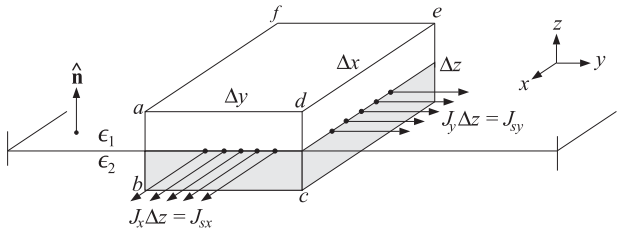
1.2 Prove the vector analysis identities:

$$\begin{aligned}
&\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi) = 0 &&\\
&\boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\psi) = \phi\nabla^2\psi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi && \text{(Green's first identity)}\\
&\boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi && \text{(Green's second identity)}\\
&\boldsymbol{\nabla}\cdot(\phi\boldsymbol{A}) = (\boldsymbol{\nabla}\phi)\cdot\boldsymbol{A} + \phi\,\boldsymbol{\nabla}\cdot\boldsymbol{A} &&\\
&\boldsymbol{\nabla}\times(\phi\boldsymbol{A}) = (\boldsymbol{\nabla}\phi)\times\boldsymbol{A} + \phi\,\boldsymbol{\nabla}\times\boldsymbol{A} &&\\
&\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) = 0 &&\\
&\boldsymbol{\nabla}\cdot\boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) &&\\
&\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} &&
\end{aligned}$$

1.3 Consider the infinitesimal volume element $\Delta x\Delta y\Delta z$ shown below, such that its upper half lies in medium ϵ_1 and its lower half in medium ϵ_2. The axes are oriented such that $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Applying the integrated form of Ampère's law to the infinitesimal face $abcd$, show that

$$H_{2y} - H_{1y} = J_x\Delta z + \frac{\partial D_x}{\partial t}\Delta z$$

In the limit $\Delta z \to 0$, the second term in the right-hand side may be assumed to go to zero, whereas the first term will be non-zero and may be set equal to a surface current density, that is, $J_{sx} \equiv \lim_{\Delta z\to 0}(J_x\Delta z)$. Show that this leads to the boundary condition $H_{1y} - H_{2y} = -J_{sx}$. Similarly, show that $H_{1x} - H_{2x} = J_{sy}$, and that these two boundary conditions can be combined vectorially into Eq. (1.5.4).



Next, apply the integrated form of Gauss's law to the same volume element and show the boundary condition: $D_{1z} - D_{2z} = \rho_s = \lim_{\Delta z\to 0}(\rho\Delta z)$.

1.4 Show that the time average of the *product* of two harmonic quantities $\mathcal{A}(t) = \text{Re}[Ae^{j\omega t}]$ and $\mathcal{B}(t) = \text{Re}[Be^{j\omega t}]$ with phasors A, B is given by:

$$\overline{\mathcal{A}(t)\mathcal{B}(t)} = \frac{1}{T}\int_0^T \mathcal{A}(t)\mathcal{B}(t)\,dt = \frac{1}{2}\text{Re}[AB^*]$$

where $T = 2\pi/\omega$ is one period. Then, show that the time-averaged values of the cross and dot products of two time-harmonic vector quantities $\boldsymbol{\mathcal{A}}(t) = \text{Re}[\boldsymbol{A}e^{j\omega t}]$ and $\boldsymbol{\mathcal{B}}(t) = \text{Re}[\boldsymbol{B}e^{j\omega t}]$ can be expressed in terms of the corresponding phasors as follows:

$$\overline{\boldsymbol{\mathcal{A}}(t)\times\boldsymbol{\mathcal{B}}(t)} = \frac{1}{2}\text{Re}[\boldsymbol{A}\times\boldsymbol{B}^*], \qquad \overline{\boldsymbol{\mathcal{A}}(t)\cdot\boldsymbol{\mathcal{B}}(t)} = \frac{1}{2}\text{Re}[\boldsymbol{A}\cdot\boldsymbol{B}^*]$$

1.5 Assuming that $\boldsymbol{B} = \mu\boldsymbol{H}$, show that Maxwell's equations (1.9.2) imply the following complex-valued version of Poynting's theorem:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{E}\times\boldsymbol{H}^*) = -j\omega\mu\boldsymbol{H}\cdot\boldsymbol{H}^* - \boldsymbol{E}\cdot\boldsymbol{J}_{\text{tot}}^*, \qquad \text{where } \boldsymbol{J}_{\text{tot}} = \boldsymbol{J} + j\omega\boldsymbol{D}$$

Extracting the real-parts of both sides and integrating over a volume V bounded by a closed surface S, show the time-averaged form of energy conservation:

$$-\oint_S \frac{1}{2}\text{Re}[\boldsymbol{E}\times\boldsymbol{H}^*]\cdot d\boldsymbol{S} = \int_V \frac{1}{2}\text{Re}[\boldsymbol{E}\cdot\boldsymbol{J}_{\text{tot}}^*]\,dV$$

which states that the net time-averaged power flowing into a volume is dissipated into heat. For a lossless dielectric, show that the above integrals are zero and provide an interpretation.

1.6 Assuming that $\boldsymbol{D} = \epsilon\boldsymbol{E}$ and $\boldsymbol{B} = \mu\boldsymbol{H}$, show that Maxwell's equations (1.1.1) imply the following relationships:

$$\rho E_x + (\boldsymbol{D}\times\frac{\partial\boldsymbol{B}}{\partial t})_x = \boldsymbol{\nabla}\cdot(\epsilon E_x\boldsymbol{E} - \hat{\mathbf{x}}\frac{1}{2}\epsilon E^2)$$
$$(\boldsymbol{J}\times\boldsymbol{B})_x + (\frac{\partial\boldsymbol{D}}{\partial t}\times\boldsymbol{B})_x = \boldsymbol{\nabla}\cdot(\mu H_x\boldsymbol{H} - \hat{\mathbf{x}}\frac{1}{2}\mu H^2)$$

where the subscript x means the x-component. From these, derive the following relationship that represents momentum conservation:

$$f_x + \frac{\partial G_x}{\partial t} = \boldsymbol{\nabla}\cdot\boldsymbol{T}_x \tag{1.19.1}$$

where f_x, G_x are the x-components of the vectors $\boldsymbol{f} = \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B}$ and $\boldsymbol{G} = \boldsymbol{D}\times\boldsymbol{B}$, and $\boldsymbol{T}_x$ is defined to be the vector (equal to Maxwell's stress tensor acting on the unit vector $\hat{\mathbf{x}}$):

$$\boldsymbol{T}_x = \epsilon E_x \boldsymbol{E} + \mu H_x \boldsymbol{H} - \hat{\mathbf{x}}\,\frac{1}{2}(\epsilon E^2 + \mu H^2)$$

Write similar equations of the y, z components. The quantity G_x is interpreted as the field momentum (in the x-direction) per unit volume, that is, the momentum density.

1.7 Show that the causal and stable time-domain dielectric response corresponding to Eq. (1.11.5) is given as follows:

$$\epsilon(t) = \epsilon_0\delta(t) + \epsilon_0\chi(t)\,, \quad \chi(t) = \frac{\omega_p^2}{\bar{\omega}_0}e^{-\gamma t/2}\sin(\bar{\omega}_0 t)\,u(t) \tag{1.19.2}$$

where $u(t)$ is the unit-step function and $\bar{\omega}_0 = \sqrt{\omega_0^2 - \gamma^2/4}$, and we must assume that $\gamma < 2\omega_0$, as is typically the case in practice. Discuss the solution for the case $\gamma/2 > \omega_0$.

1.8 Show that the plasma frequency for electrons can be expressed in the simple numerical form: $f_p = 9\sqrt{N}$, where f_p is in Hz and N is the electron density in electrons/m^3. What is f_p for the ionosphere if $N = 10^{12}$? [*Ans.* 9 MHz.]

1.9 Show that the relaxation equation (1.13.2) can be written in the following form in terms of the dc-conductivity σ defined by Eq. (1.12.4):

$$\frac{1}{\gamma}\ddot{\rho}(\mathbf{r},t) + \dot{\rho}(\mathbf{r},t) + \frac{\sigma}{\epsilon_0}\rho(\mathbf{r},t) = 0$$

Then, show that it reduces to the naive relaxation equation (1.7.3) in the limit $\tau = 1/\gamma \to 0$. Show also that in this limit, Ohm's law (1.13.1) takes the instantaneous form $\boldsymbol{J} = \sigma\boldsymbol{E}$, from which the naive relaxation constant $\tau_{\text{relax}} = \epsilon_0/\sigma$ was derived.

1.10 Conductors and plasmas exhibit anisotropic and birefringent behavior when they are in the presence of an external magnetic field. The equation of motion of conduction electrons in a constant external magnetic field is $m\dot{\boldsymbol{v}} = e(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) - m\gamma\boldsymbol{v}$, with the collisional term included. Assume the magnetic field is in the z-direction, $\boldsymbol{B} = \hat{\mathbf{z}}\,B$, and that $\boldsymbol{E} = \hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y$ and $\boldsymbol{v} = \hat{\mathbf{x}}v_x + \hat{\mathbf{y}}v_y$.

a. Show that in component form, the above equations of motion read:

$$\begin{aligned}\dot{v}_x &= \frac{e}{m}E_x + \omega_B v_y - \gamma v_x \\ \dot{v}_y &= \frac{e}{m}E_y - \omega_B v_x - \gamma v_y\end{aligned} \qquad \text{where} \quad \omega_B = \frac{eB}{m} = \text{(cyclotron frequency)}$$

What is the cyclotron frequency in Hz for electrons in the Earth's magnetic field B = 0.4 gauss = 0.4×10^{-4} Tesla? [*Ans.* 1.12 MHz.]

b. To solve this system, work with the combinations $v_x \pm jv_y$. Assuming harmonic time-dependence, show that the solution is:

$$v_x \pm jv_y = \frac{\frac{e}{m}(E_x \pm jE_y)}{\gamma + j(\omega \pm \omega_B)}$$

c. Define the induced currents as $\boldsymbol{J} = Ne\boldsymbol{v}$. Show that:

$$J_x \pm jJ_y = \sigma_\pm(\omega)\,(E_x \pm jE_y)\,, \qquad \text{where} \quad \sigma_\pm(\omega) = \frac{\gamma\sigma_0}{\gamma + j(\omega \pm \omega_B)}$$

where $\sigma_0 = \dfrac{Ne^2}{m\gamma}$ is the dc value of the conductivity.

d. Show that the t-domain version of part (c) is:

$$J_x(t) \pm jJ_y(t) = \int_0^t \sigma_\pm(t-t')\,\big(E_x(t') \pm jE_y(t')\big)\,dt'$$

where $\sigma_\pm(t) = \gamma\sigma_0 e^{-\gamma t}e^{\mp j\omega_B t}u(t)$ is the inverse Fourier transform of $\sigma_\pm(\omega)$ and $u(t)$ is the unit-step function.

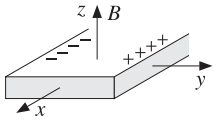
e. Rewrite part (d) in component form:

$$\begin{aligned}J_x(t) &= \int_0^t \big[\sigma_{xx}(t-t')E_x(t') + \sigma_{xy}(t-t')E_y(t')\big]\,dt' \\ J_y(t) &= \int_0^t \big[\sigma_{yx}(t-t')E_x(t') + \sigma_{yy}(t-t')E_y(t')\big]\,dt'\end{aligned}$$

and identify the quantities $\sigma_{xx}(t), \sigma_{xy}(t), \sigma_{yx}(t), \sigma_{yy}(t)$.

f. Evaluate part (e) in the special case $E_x(t) = E_x u(t)$ and $E_y(t) = E_y u(t)$, where E_x, E_y are constants, and show that after a long time the steady-state version of part (e) will be:

$$\begin{aligned}J_x &= \sigma_0\frac{E_x + bE_y}{1+b^2} \\ J_y &= \sigma_0\frac{E_y - bE_x}{1+b^2}\end{aligned}$$



where $b = \omega_B/\gamma$. If the conductor has finite extent in the y-direction, as shown above, then no steady current can flow in this direction, $J_y = 0$. This implies that if an electric field is applied in the x-direction, an electric field will develop across the y-ends of the conductor, $E_y = bE_x$. The conduction charges will tend to accumulate either on the right or the left side of the conductor, depending on the sign of b, which depends on the sign of the electric charge e. This is the *Hall effect* and is used to determine the sign of the conduction charges in semiconductors, e.g., positive holes for p-type, or negative electrons for n-type.

What is the numerical value of b for electrons in copper if B is 1 gauss? [*Ans.* 4.3×10^{-7}.]

g. For a collisionless plasma ($\gamma = 0$), show that its dielectric behavior is determined from $D_x \pm jD_y = \epsilon_\pm(\omega)\,(E_x \pm jE_y)$, where

$$\epsilon_\pm(\omega) = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega\,(\omega \pm \omega_B)}\right)$$

where ω_p is the plasma frequency. Thus, the plasma exhibits birefringence.

1.11 This problem deals with various properties of the Kramers-Kronig dispersion relations for the electric susceptibility, given by Eq. (1.17.8).

a. Using the symmetry properties $\chi_r(\omega) = \chi_r(-\omega)$ and $\chi_i(\omega) = -\chi_i(-\omega)$, show that (1.17.8) can be written in the folded form of Eq. (1.17.9).

b. Using the definition of principal-value integrals, show the following integral:

$$\mathcal{P}\int_0^\infty \frac{d\omega'}{\omega'^2 - \omega^2} = 0 \tag{1.19.3}$$

Hint: You may use the following indefinite integral: $\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$.

c. Using Eq. (1.19.3), show that the relations (1.17.9) may be rewritten as ordinary integrals (without the $\mathcal{P}$ instruction) as follows:

$$\begin{aligned}\chi_r(\omega) &= \frac{2}{\pi}\int_0^\infty \frac{\omega'\chi_i(\omega')-\omega\chi_i(\omega)}{\omega'^2-\omega^2}\,d\omega' \\ \chi_i(\omega) &= -\frac{2}{\pi}\int_0^\infty \frac{\omega\chi_r(\omega')-\omega\chi_r(\omega)}{\omega'^2-\omega^2}\,d\omega'\end{aligned} \tag{1.19.4}$$

Hint: You will need to argue that the integrands have no singularity at $\omega' = \omega$.

d. For a simple oscillator model of dielectric polarization, the susceptibility is given by:

$$\begin{aligned}\chi(\omega) &= \chi_r(\omega)-j\chi_i(\omega) = \frac{\omega_p^2}{\omega_0^2-\omega^2+j\gamma\omega} \\ &= \frac{\omega_p^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} - j\frac{\gamma\omega\omega_p^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\end{aligned} \tag{1.19.5}$$

Show that for this model the quantities $\chi_r(\omega)$ and $\chi_i(\omega)$ satisfy the modified Kramers-Kronig relationships (1.19.4). *Hint*: You may use the following definite integrals, for which you may assume that $0 < \gamma < 2\omega_0$:

$$\frac{2}{\pi}\int_0^\infty \frac{dx}{(\omega_0^2-x^2)^2+\gamma^2x^2} = \frac{1}{\gamma\omega_0^2}, \qquad \frac{2}{\pi}\int_0^\infty \frac{x^2dx}{(\omega_0^2-x^2)^2+\gamma^2x^2} = \frac{1}{\gamma}$$

Indeed, show that these integrals may be reduced to the following ones, which can be found in standard tables of integrals:

$$\frac{2}{\pi}\int_0^\infty \frac{dy}{1-2y^2\cos\theta+y^4} = \frac{2}{\pi}\int_0^\infty \frac{y^2dy}{1-2y^2\cos\theta+y^4} = \frac{1}{\sqrt{2(1-\cos\theta)}}$$

where $\sin(\theta/2) = \gamma/(2\omega_0)$.

e. Consider the limit of Eq. (1.19.5) as $\gamma \to 0$. Show that in this case the functions χ_r, χ_i are given as follows, and that they still satisfy the Kramers-Kronig relations:

$$\chi_r(\omega) = \mathcal{P}\frac{\omega_p^2}{\omega_0-\omega} + \mathcal{P}\frac{\omega_p^2}{\omega_0+\omega}, \quad \chi_i(\omega) = \frac{\pi\omega_p^2}{2\omega_0}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$$

1.12 Derive the Kramers-Kronig relationship of Eq. (1.17.7) by starting with the causality condition $\chi(t)\,u(-t) = 0$ and translating it to the frequency domain, that is, expressing it as the convolution of the Fourier transforms of $\chi(t)$ and $u(-t)$.

1.13 An isotropic homogeneous lossless dielectric medium is moving with uniform velocity $\boldsymbol{v}$ with respect to a fixed coordinate frame S. In the frame S' moving with dielectric, the constitutive relations are assumed to be the usual ones, that is, $\boldsymbol{D}' = \epsilon\boldsymbol{E}'$ and $\boldsymbol{B}' = \mu\boldsymbol{H}'$. Using the Lorentz transformations given in Eq. (H.30) of Appendix H, show that the constitutive relations take the following form in the fixed frame S:

$$\boldsymbol{D} = \epsilon\boldsymbol{E} + a\boldsymbol{v}\times(\boldsymbol{H}-\epsilon\boldsymbol{v}\times\boldsymbol{E}), \quad \boldsymbol{B} = \mu\boldsymbol{H} - a\boldsymbol{v}\times(\boldsymbol{E}+\mu\boldsymbol{v}\times\boldsymbol{H}), \quad a = \frac{\epsilon\mu-\epsilon_0\mu_0}{1-\epsilon\mu v^2}$$